Session #10: (More) Trapdoors and Applications

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Winter School on Lattice-Based Cryptography and Applications

Bar-Ilan University, Israel

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Lattice-Based One-Way Functions

- Public key $\begin{bmatrix} \cdots & \mathbf{A} & \cdots \end{bmatrix} \in \mathbb{Z}_q^{n \times m}$ for $q = \mathsf{poly}(n)$, $m = \Omega(n \log q)$.

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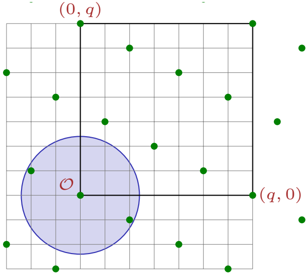

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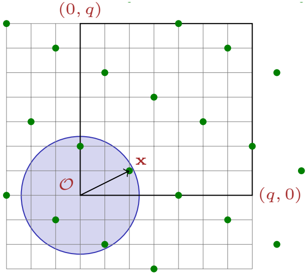

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- $f_{\mathbf{A}}$, $g_{\mathbf{A}}$ in forward direction yield CRHFs, CPA security (w/FHE!)

  ...but not much else.

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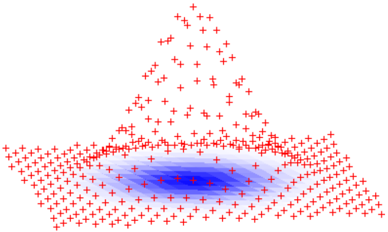

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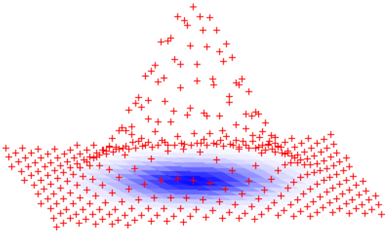
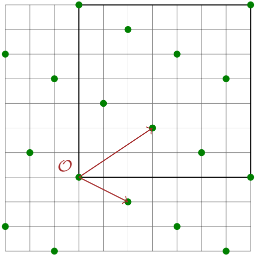
- How? Use a "strong trapdoor" for $\mathbf{A}$: a short basis of $\Lambda^\perp(\mathbf{A})$

[Babai'86,GGH'97,Klein'01,GPV'08,P'10]





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Other "Black-Box" Applications of f^{-1}, g^{-1}

- Standard Model (no RO) signatures [CHKP'10,R'10,B'10]
- SM CCA-secure encryption [PW'08,P'09]
- SM (Hierarchical) IBE [GPV'08,CHKP'10,ABB'10a,ABB'10b]
- Many more: OT, NISZK, homom enc/sigs, deniable enc, func enc, ...
  [PVW'08,PV'08,GHV'10,GKV'10,BF'10a,BF'10b,OPW'11,AFV'11,ABVVW'11,...]

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	tight, iterative, fp	looser, parallel, offline
$g_{\mathbf{A}}^{-1}$	[Babai'86]	[Babai'86]
$f_{\mathbf{A}}^{-1}$	[Klein'01,GPV'08]	[P'10]

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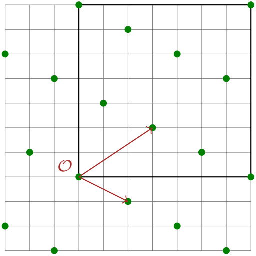

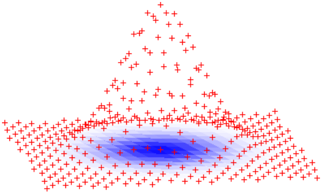

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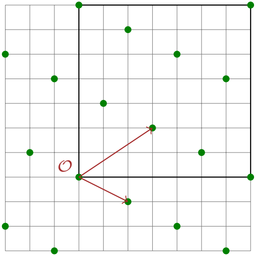

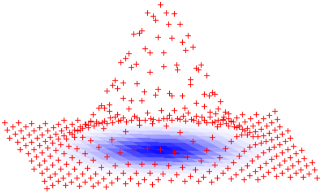

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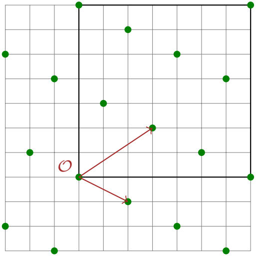

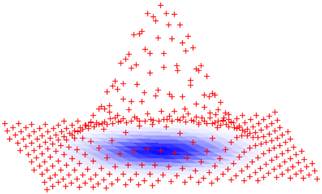

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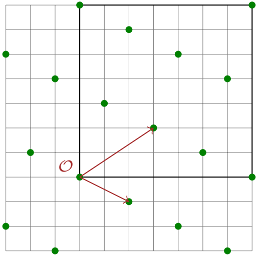

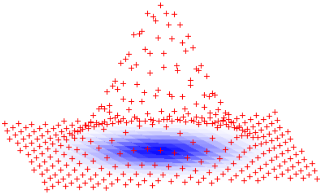

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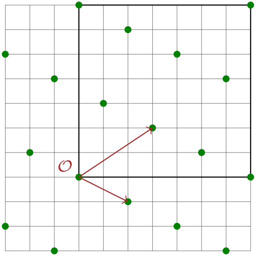

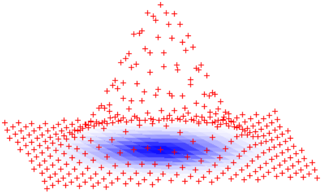

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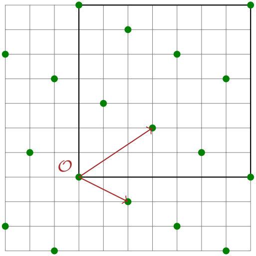

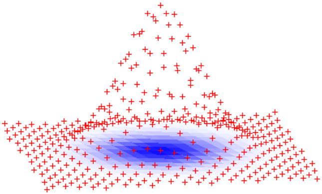

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✔ Better dimension m & quality s

$\implies$ "win-win-win" in security-keysize-runtime

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- ✔ New kind of trapdoor — not a basis! (But just as powerful.)
- ✔ More efficient applications: CCA, (H)IBE in standard model

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3. Reduce $f_{\mathbf{A}}^{-1}$, $g_{\mathbf{A}}^{-1}$ to $f_{\mathbf{G}}^{-1}$, $g_{\mathbf{G}}^{-1}$ plus pre-/post-processing.

Step 1: Gadget $\mathbf{G}$ and Inversion Algorithms

- Let $q = 2^k$. Define 1-by-k "parity check" vector

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- To sample Gaussian preimage for $u = f_{\mathbf{g}}(\mathbf{x}) := \langle \mathbf{g}, \mathbf{x} \rangle$:

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- To invert LWE function $g_{\mathbf{g}} \colon \mathbb{Z}_q \times \mathbb{Z}^k \to \mathbb{Z}_q^k$:

$$s \cdot \mathbf{g} + \mathbf{e} = \begin{bmatrix} s + e_0 & 2s + e_1 & \cdots & 2^{k-1}s + e_{k-1} \end{bmatrix} \bmod q.$$

  - ⋆ Get lsb(s) from $2^{k-1}s + e_{k-1}$. Then get next bit of s, etc.
    Works exactly when every $e_i \in [-\frac{q}{4}, \frac{q}{4})$.
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  Also applies to $\mathbf{H} \cdot \mathbf{G}$ for any invertible $\mathbf{H} \in \mathbb{Z}_q^{n \times n}$.

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1. Define semi-random $[\bar{\mathbf{A}} \mid \mathbf{G}]$ for uniform $\bar{\mathbf{A}} \in \mathbb{Z}_q^{n \times \bar{m}}$.

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- ⋆ $[\mathbf{I} \mid \bar{\mathbf{A}} \mid -(\bar{\mathbf{A}}\mathbf{R}_1 + \mathbf{R}_2)]$ is pseudorandom (under LWE) for $\bar{m} = n$.

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Relating New and Old Trapdoors

Given a basis $\mathbf{S}$ for $\Lambda^{\perp}(\mathbf{G})$ and a trapdoor $\mathbf{R}$ for $\mathbf{A}$,
we can efficiently construct a basis $\mathbf{S}_{\mathbf{A}}$ for $\Lambda^{\perp}(\mathbf{A})$
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(But we'll never need to.)

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Given $\mathbf{b}^t = \mathbf{s}^t\mathbf{A} + \mathbf{e}^t$, recover $\mathbf{s}$ from

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Works if each entry of $\mathbf{e}^t\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]$ in $[-\frac{q}{4}, \frac{q}{4}) \Leftarrow \|\mathbf{e}\| < q/(4s_1(\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]))$.

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Given $\mathbf{u}$, sample $\mathbf{z} \leftarrow f_{\mathbf{G}}^{-1}(\mathbf{u})$ and output $\mathbf{x} = \left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]\mathbf{z} \in f_{\mathbf{A}}^{-1}(\mathbf{u})$?

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- Solution: use offline 'perturbation' [P'10] to get spherical Gaussian w/ std dev $\approx s_1(\mathbf{R})$: output $\mathbf{x} = \mathbf{p} + \left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right]\mathbf{z}$.

Application: Efficient IBE *a la* [ABB'10]

- Setup: choose $\mathbf{A} = [\bar{\mathbf{A}} \mid -\bar{\mathbf{A}}\mathbf{R}]$. Let $mpk = (\mathbf{A}, \mathbf{u})$, $msk = \mathbf{R}$.
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- Encrypt to $\mathbf{A}_{id}$, decrypt using sk_{id} as in 'dual' system [GPV'08].

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$$\mathbf{A}_{id} = \mathbf{A} + [\mathbf{0} \mid \mathbf{H}_{id} \cdot \mathbf{G}] = [\bar{\mathbf{A}} \mid \mathbf{H}_{id} \cdot \mathbf{G} - \bar{\mathbf{A}}\mathbf{R}].$$

- Encrypt to $\mathbf{A}_{id}$, decrypt using sk_{id} as in 'dual' system [GPV'08].

- Security ("puncturing"): Given target id^* (selective security), set up

$$\mathbf{A} = [\bar{\mathbf{A}} \mid -\mathbf{H}_{id^*} \cdot \mathbf{G} - \bar{\mathbf{A}}\mathbf{R}] \Longrightarrow \mathbf{A}_{id} = [\bar{\mathbf{A}} \mid (\mathbf{H}_{id} - \mathbf{H}_{id^*})\mathbf{G} - \bar{\mathbf{A}}\mathbf{R}].$$

  - ⋆ $\mathbf{H}_{id} - \mathbf{H}_{id^*}$ is invertible for all $id \neq id^*$, so can extract sk_{id} using $\mathbf{R}$.
  - ⋆ $\mathbf{A}_{id^*} = [\bar{\mathbf{A}} \mid -\bar{\mathbf{A}}\mathbf{R}]$, so can embed an LWE challenge at id^*.

Trapdoor Delegation [CHKP'10]

- Suppose $\mathbf{R}$ is a trapdoor for $\mathbf{A}$, i.e. $\mathbf{A}\left[\begin{smallmatrix}\mathbf{R}\\ \mathbf{I}\end{smallmatrix}\right] = \mathbf{H} \cdot \mathbf{G}$.

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$$[\mathbf{A} \mid \mathbf{A}']\left[\begin{smallmatrix}\mathbf{R}'\\ \mathbf{I}\end{smallmatrix}\right] = \mathbf{H}' \cdot \mathbf{G} \iff \mathbf{A}\mathbf{R}' = \mathbf{H}' \cdot \mathbf{G} - \mathbf{A}'.$$

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- Note: $\mathbf{R}'$ is only width$(\mathbf{A}) \times$ width$(\mathbf{G}) = m \times n\log q$.

  So size of $\mathbf{R}'$ grows only as $O(m)$, not $\Omega(m^2)$ like a basis does

  Also computationally efficient: $n\log q$ samples, no HNF or ToBasis.

Hierarchical IBE [CHKP'10,ABB'10]

- Setup(d): choose $\mathbf{A}_0, \ldots, \mathbf{A}_d$ where $\mathbf{A}_\varepsilon = [\mathbf{A}_0 \mid \mathbf{A}_1]$ has trapdoor $\mathbf{R}_\varepsilon$ for tag $\mathbf{0}$. Let $msk = sk_\varepsilon = \mathbf{R}_\varepsilon$ and $mpk = \{\mathbf{A}_i\}$.

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- Extract(id): map $id = (id_1, \ldots, id_t) \mapsto (\mathbf{H}_{id_1}, \ldots \mathbf{H}_{id_t})$ (invertible). Let
$$\mathbf{A}_{id} = [\mathbf{A}_0 \mid \mathbf{A}_1 + \mathbf{H}_{id_1}\mathbf{G} \mid \cdots \mid \mathbf{A}_t + \mathbf{H}_{id_t}\mathbf{G} \mid \mathbf{A}_{t+1}].$$

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  Using sk_{id}, can delegate any $sk_{id'}$ for any nontrivial extension id'.

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- Encrypt to $\mathbf{A}_{id}$, decrypt using $\mathbf{R}_{id}$ as in [GPV'08].

- Security ("puncturing"): Set up mpk, trapdoor $\mathbf{R}$ with tags $= -id^*$.

Conclusions

- A simple trapdoor that's easy to generate, use, and understand:

  Applications made easy, end-to-end!

- Key sizes and algorithms for "strong" trapdoors are now realistic

Selected bibliography for this talk:

CHKP'10 D. Cash, D. Hofheinz, E. Kiltz, C. Peikert, "Bonsai Trees, or How to Delegate a Lattice Basis," Eurocrypt'10 / J. Crypt'11.

ABB'10 S. Agrawal, D. Boneh, X. Boyen, "Efficient Lattice (H)IBE in the Standard Model," Eurocrypt'10.

MP'12 D. Micciancio, C. Peikert, "Trapdoors for Lattices: Simpler, Tighter, Faster, Smaller," Eurocrypt'12.