Crash Course in Quantum Computing

Hour 3: Advanced Quantum Information Theory

BIU Winter School on Cryptography 2021

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Mixed States

Probabilistic mixtures of pure quantum states

- Up till now, we've represented quantum states as unit vectors in $\mathbb{C}^d$. These are called ***pure states***.
- Describing a quantum system using a pure state $|\psi\rangle$ indicates that state of the system is ***determined***.
- **Ex**: taking a qubit in the $|0\rangle$ state, and applying H to it.
- What if someone flips a coin and hands you either $|0\rangle$ or $|+\rangle$ depending on the coin? If you do not see the coin, then the state given to you is a ***mixed state***. We can describe this as a probabilistic mixture:

$$\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |+\rangle\right)$$

Density matrices

- A d-dimensional density matrix is a matrix $\rho \in \mathbb{C}^{d\times d}$ such that
  - ρ is positive semidefinite
  - $Tr(\rho) = 1$

sum up diagonal entries.

- Density matrices describe mixed states.
- A pure state $|\psi\rangle \in \mathbb{C}^d$ corresponds to density matrix $|\psi\rangle\langle\psi|$.
- A mixture $\{(p_1, |\psi_1\rangle), \ldots, (p_k, |\psi_k\rangle)\}$ corresponds to density matrix $\sum_i p_i\, |\psi_i\rangle\langle\psi_i|$

$$Tr\Big(\sum_i p_i\, |\psi_i\rangle\langle\psi_i|\Big) = \sum_i p_i \cdot \underbrace{Tr(|\psi_i\rangle\langle\psi_i|)}_{=1} = \sum p_i = 1.$$

Density matrices

- **Ex**: $|0\rangle, |1\rangle$

$$|0\rangle\langle 0| \qquad |1\rangle\langle 1|$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

- **Ex:** $\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |+\rangle\right)$

$$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|+\rangle\langle +|$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

Density matrices

- **Ex:** $\left(\frac{1}{2}, |0\rangle\right), \left(\frac{1}{2}, |1\rangle\right)$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2}\cdot I$$

maximally mixed state.

- **Ex:** $\left(\frac{1}{2}, |+\rangle\right), \left(\frac{1}{2}, |-\rangle\right)$

$$= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} = \frac{1}{2} I$$

Projective measurements

$M = \{M_1, M_2, \ldots, M_k\}$ is a k-outcome projective measurement if

- Each M_i is a Hermitian projection matrix, i.e., $M_i^\dagger = M_i$ and $M_i^2 = M_i$
- $M_1 + M_2 + \cdots + M_k = I$

Measuring a pure state $|\psi\rangle$ using M yields

- outcome i with probability $\|M_i|\psi\rangle\|^2$
- Post-measurement state $\frac{M_i|\psi\rangle}{\|M_i|\psi\rangle\|^2}$

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Ex: measuring according to orthonormal basis $B = \{|b_0\rangle, \ldots, |b_{d-1}\rangle\}$ corresponds to projectors $M_i = |b_i\rangle\langle b_i|$

Density matrices

- Density matrices encode everything that is physically relevant about a probabilistic mixture of pure states.

- **Unitary evolution**: $\rho \mapsto U\rho U^\dagger$

$\rho = |\psi\rangle\langle\psi|$ $\quad\quad |\psi\rangle \mapsto U|\psi\rangle.$

$\mapsto U|\psi\rangle\langle\psi|U^\dagger.$

- **Measurement**: Let $M = \{M_1, M_2, \dots, M_k\}$ denote a k-outcome projective measurement. Then measuring ρ with M yields outcome i with probability $Tr(M_i\, \rho)$

matrix.

$\sum_i Tr(M_i \rho) = Tr\left(\left(\sum_i M_i\right) \cdot \rho\right)$

$= Tr(\rho) = 1.$

- **Post-measurement state**: $\rho \mapsto \frac{M_i \rho M_i}{Tr(M_i\, \rho)}$

Density matrices

- **Ex:** $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $\rho = |\psi\rangle\langle\psi|$, measure using standard basis. $M = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

$|\psi\rangle\langle\psi| = |\alpha|^2 |0\rangle\langle 0| + \alpha\beta^* |0\rangle\langle 1| + \beta\alpha^* |1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$.

$\Pr[0 \text{ outcome}] = \mathrm{Tr}(|0\rangle\langle 0| \cdot |\psi\rangle\langle\psi|) = |\alpha|^2$

$\Pr(1 \text{ outcome}] = |\beta|^2$.

- **Ex:** $\rho = \frac{I}{2}$, measure using basis $B = \{|b_0\rangle, |b_1\rangle\}$

$\Pr[|b_0\rangle \text{ outcome}] = \mathrm{Tr}(|b_0\rangle\langle b_0| \cdot \frac{I}{2}) = \frac{1}{2}\mathrm{Tr}(|b_0\rangle\langle b_0|) = \frac{1}{2}$.

$= \frac{1}{2}$.

$\Pr[|b_1\rangle \text{ outcome}] = \ldots$

Quantum One-Time Pad

- **Classical one-time pad**: Fix message $m \in \{0,1\}^n$. Let s be uniformly random n-bit string. Marginal distribution of $m \oplus s$ is uniformly random.

$Z^0 = I \quad Z^1 = Z$

$X^0 = I \quad X^1 = X.$

Quantum One-Time Pad

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- **Quantum one-time pad**: Fix qubit $|\psi\rangle \in \mathbb{C}^2$. Sample uniformly random bits $a, b \in \{0,1\}$. Apply $Z^a X^b$ to $|\psi\rangle$.

- The ensemble $\left\{\left(\frac{1}{4}, Z^a X^b |\psi\rangle\right)\right\}$ looks uniformly random.

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QOTP keys.

- The ensemble $\left\{\left(\frac{1}{4}, Z^a X^b |\psi\rangle\right)\right\}$ looks uniformly random.

- Corresponding density matrix:

$$\frac{1}{4}(|\psi\rangle\langle\psi| + X|\psi\rangle\langle\psi|X + Z|\psi\rangle\langle\psi|Z + ZX|\psi\rangle\langle\psi|XZ) = \frac{I}{2}$$

Density matrices of multiple systems

independent !!!

- Given two quantum systems described by density matrices ρ, σ, their joint system is described by the density matrix $\rho \otimes \sigma$.

- n copies of ρ is abbreviated $\rho^{\otimes n}$

If $\rho = \frac{I}{2}$, then $\rho^{\otimes n} = \frac{I_{2^n}}{2^n}$ } identity on 2^n-dim.

- Not all density matrices on multiple systems can be written as $\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots$.

- But doesn't mean entangled! For example, $\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|$ is a mixture of classical states; has *classical correlations*.

$\underbrace{\rho_{AB}}_{\text{2 qubit density matrix}} \neq \sigma \otimes \tau$. $\not\Rightarrow$ ρ is entangled.

A B

ρ_{AB}

Traces and partial traces

- $Tr(\rho \otimes \sigma) = Tr(\rho) \cdot Tr(\sigma)$

- Given density matrix ρ_{AB} on systems AB, can obtain density matrix on system A only via the ***partial trace***:

2 qubit density.

$$\underbrace{\rho_A}_{\text{1-qubit density}} = \underbrace{Tr_B}_{\text{matrix.}}(\rho_{AB})$$

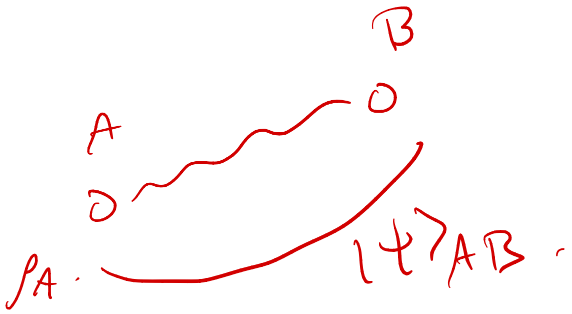
- $Tr_B(\cdot)$ denotes "tracing out" (a.k.a. marginalizing over) the B subsystem.

- Partial trace $Tr_B(\cdot)$ defined as $Tr_B(|a_1, b_1\rangle\langle a_2, b_2|) = \cancel{\langle a_2|a_1\rangle \cdot |b_1\rangle\langle b_2|}$ for all vectors $|a_1\rangle, |a_2\rangle, |b_1\rangle, |b_2\rangle$.

outer product in space $A \otimes B$.

$= \underbrace{|a_1\rangle\langle a_2|}_{\text{outer product in space } A.} \cdot \underbrace{\langle b_2|b_1\rangle}_{\text{number}}$

Traces and partial traces



- Every mixed state ρ_A on a system A is also the result of taking a partial trace of a pure state $|\psi\rangle_{AB}$ on systems AB:

$$\rho_A = Tr_B(|\psi\rangle\langle\psi|_{AB})$$

- Such a pure state $|\psi\rangle$ is called a ***purification*** of ρ.

- Purifications of density matrices are not unique.

Density matrices

- **Ex:** $\rho = |0\rangle\langle 0| \otimes |+\rangle\langle +|$

$\mathrm{Tr}_B(\rho) = |0\rangle\langle 0|$

$\mathrm{Tr}_A(\rho) = |+\rangle\langle +|.$

- **Ex:** $\rho = |EPR\rangle\langle EPR|$

$\mathrm{Tr}_A(\rho) = \mathrm{Tr}_B(\rho) = \frac{I}{2}.$

Distinguishability of density matrices

Given two density matrices ρ and σ of the same dimension, we can measure how close they are via the ***trace distance***:

quantum analogue of total variation distance.

$$D(\rho,\sigma) = \frac{1}{2}\|\rho - \sigma\|_1 = \frac{1}{2}Tr(|\rho - \sigma|)$$

Operational meaning: Trace distance $D(\rho,\sigma)$ is equivalently defined as maximum probability of distinguishing between ρ,σ using ANY possible quantum operation (measurements or unitaries).

Distinguishability of density matrices

Nice properties:

1. Nonnegative: $D(\rho, \sigma) \geq 0$, and achieves 0 if and only if $\rho = \sigma$.

2. Symmetric: $D(\rho, \sigma) = D(\sigma, \rho)$

3. Triangle inequality: $D(\rho, \sigma) \leq D(\rho, \tau) + D(\tau, \sigma)$

4. Convex: $D(\sum_i p_i \, \rho_i, \sigma) \leq \sum_i p_i \, D(\rho_i, \sigma)$

5. Does not increase when tracing out systems: $D(\rho_A, \sigma_A) \leq D(\rho_{AB}, \sigma_{AB})$

6. Unitarily invariant: $D\left(U\rho U^\dagger, U\sigma U^\dagger\right) = D(\rho, \sigma)$

Density matrices

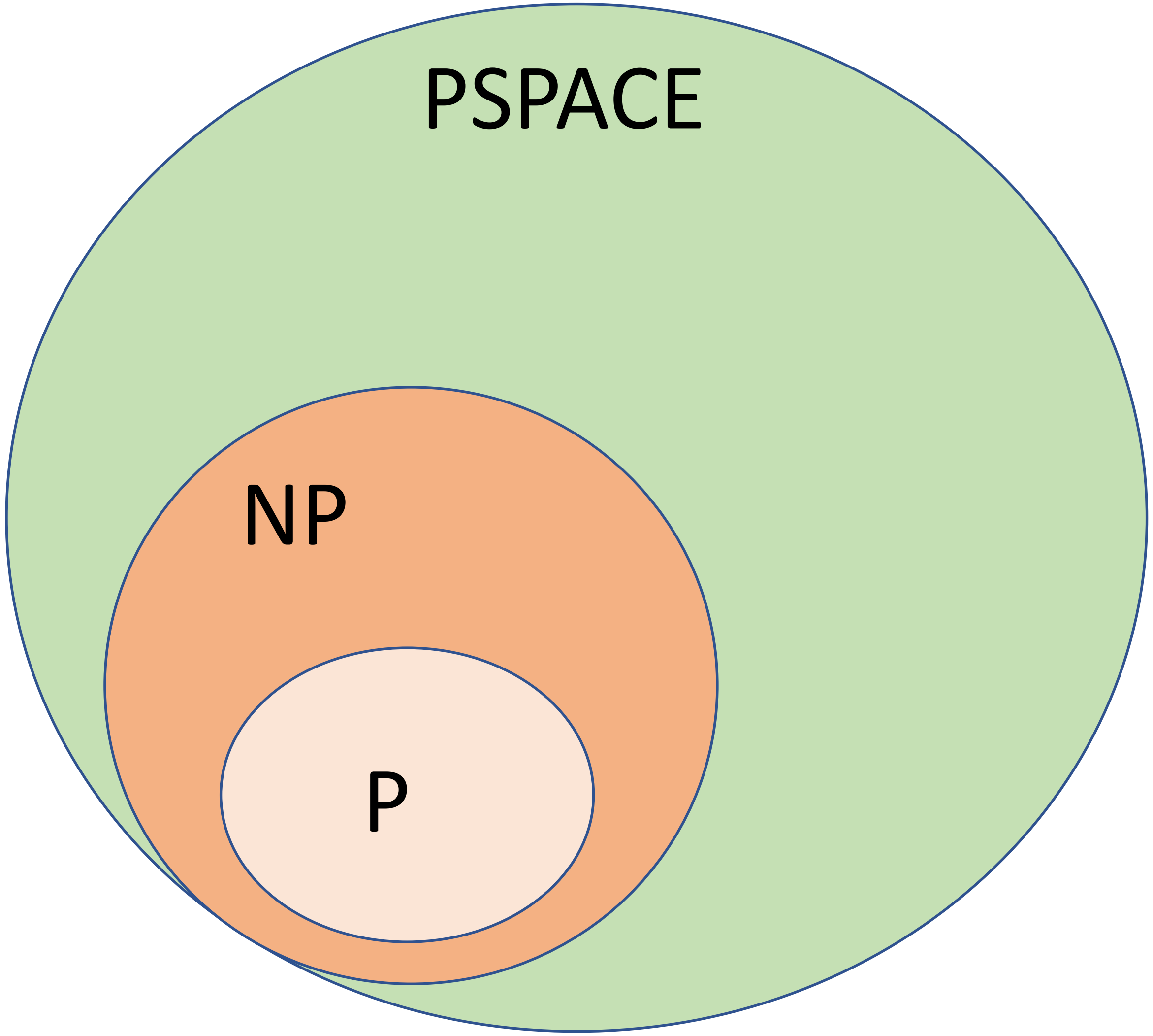
- **Ex:** $\rho = \frac{1}{2}|0,0\rangle\langle 0,0| + \frac{1}{2}|1,1\rangle\langle 1,1|$ $\sigma = \frac{1}{2}|0,1\rangle\langle 0,1| + \frac{1}{2}|1,0\rangle\langle 1,0|$

$D(\rho_{AB}, \sigma_{AB}) = 1.$

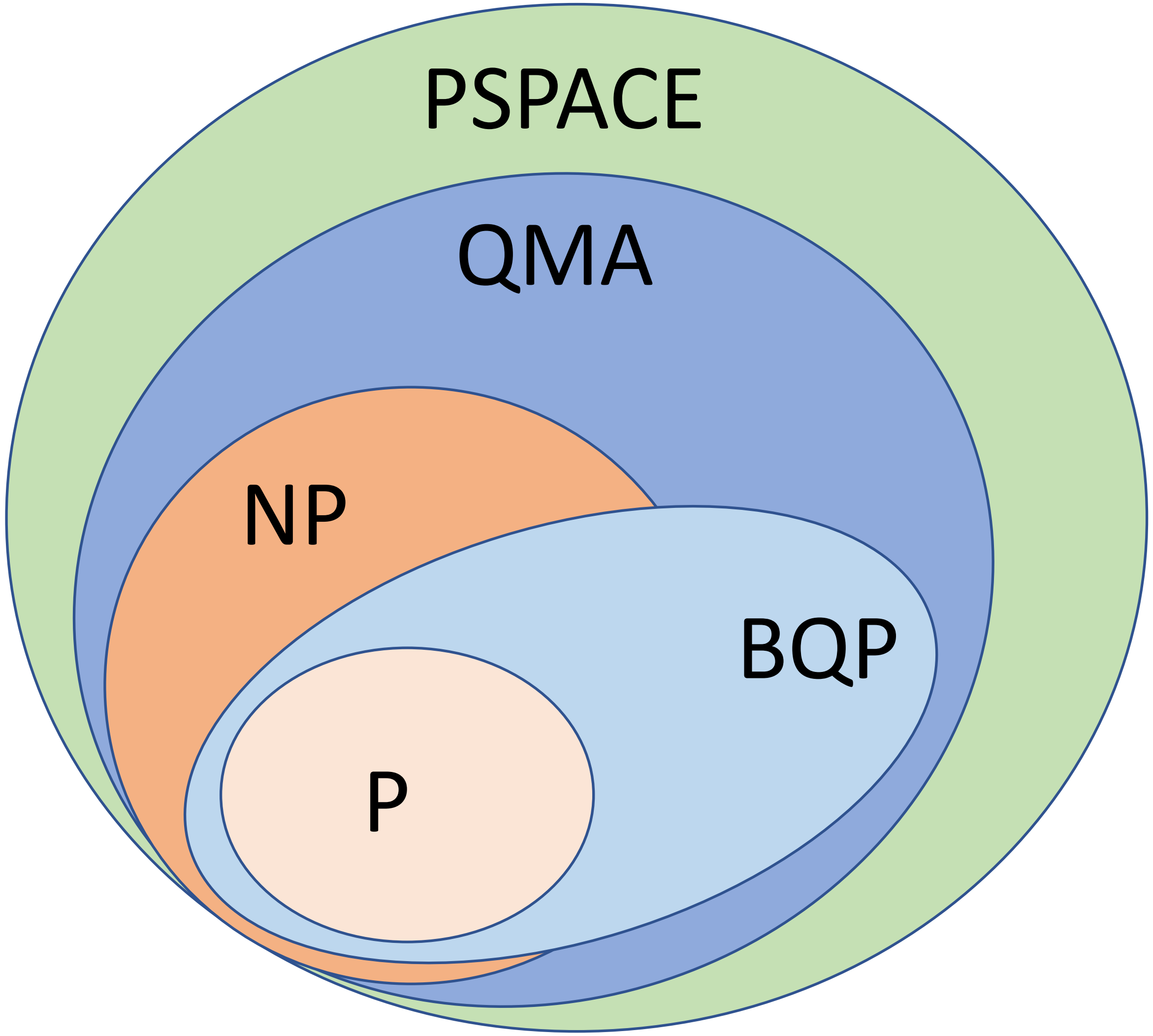
$D(\rho_A, \sigma_A) = 0.$

Quantum Complexity Theory

The Complexity Zoo



The Complexity Zoo

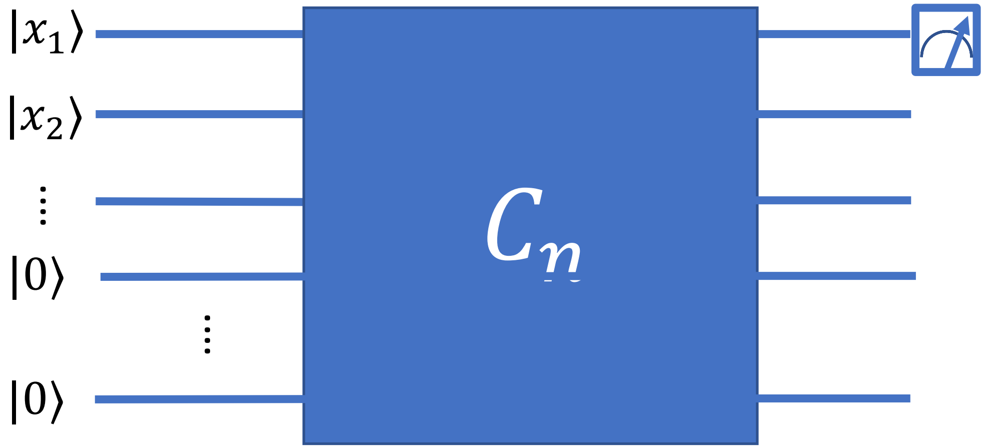


BQP

Language $L \subseteq \{0,1\}^*$ is in ***Bounded-Error Quantum Polynomial Time*** (***BQP***) if there exist a family of circuits $\{C_1, C_2, \ldots\}$ that are uniformly generated and satisfy:

- $|C_n| \leq O(n^c)$
- For all $x \in \{0,1\}^n$
  - If $x \in L \Longrightarrow \Pr[C_n \; accepts \; x] \geq \frac{2}{3}$ (Completeness)
  - If $x \notin L \Longrightarrow \Pr[C_n \; accepts \; x] \leq \frac{1}{3}$ (Soundness)

wlog assume completeness and soundness errors are $\exp(-\Omega(n))$; by repeating circuit poly(n) times and taking MAJ.



BQP

Problems in BQP:

- All problems in BPP
- Factoring, Discrete Logarithm
- Simulating quantum systems.

Canonical BQP-complete (promise) problem:

QCIRCUIT: given classical description of quantum circuit C, decide whether C accepts on the all zeroes input with probability at least $\frac{2}{3}$ or at most $\frac{1}{3}$.

QMA = quantum analogue of NP (or MA).

Language $L \subseteq \{0,1\}^*$ is in ***Quantum Merlin-Arthur*** (***QMA***) if there exist a family of *verifier* circuits $\{C_1, C_2, \dots\}$ that are uniformly generated and satisfy:

- $|C_n| \leq n^c$
- For all $x \in \{0,1\}^n$
  - If $x \in L \implies \exists |\psi\rangle, \Pr[C_n\ accepts\ |x\rangle \otimes |\psi\rangle] \geq \frac{2}{3}$ (Completeness)
  - If $x \notin L \implies \forall |\psi\rangle, \Pr[C_n\ accepts\ |x\rangle \otimes |\psi\rangle] \leq \frac{1}{3}$ (Soundness)

wlog, can assume completeness/soundness error is exponentially small. (Marriott-Watrous amplification),

QMA

Problems in QMA:

- All problems in BQP
- All problems in NP
- Finding minimum energy states of quantum systems (the Local Hamiltonians problem)

Canonical QMA-complete (promise) problem:

Q-VER-CIRCUIT: given classical description of quantum circuit C, decide if

- There exists a quantum state $|\psi\rangle$ such that C accepts $|\psi\rangle \otimes |0 \cdots 0\rangle$ with probability at least $\frac{2}{3}$, or
- All states $|\psi\rangle$ are accepted with probability at most $\frac{1}{3}$.

Local Hamiltonians problem

(k, α, β)-**Local Hamiltonians problem**: given classical description of measurements $\{H_1, H_2, \ldots, H_m\}$ on n qubits where each H_i

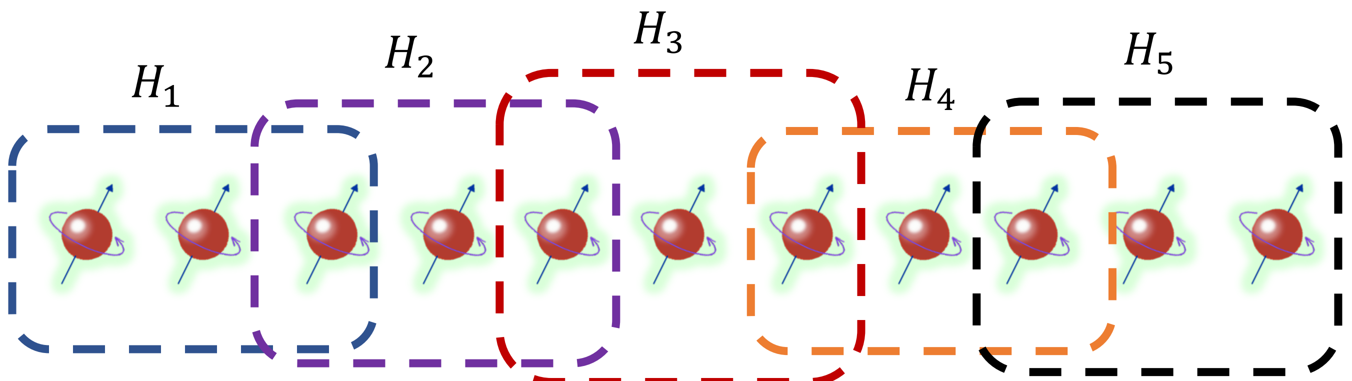
- Acts on k qubits
- Is a two-outcome measurement (with outcomes labelled "Accept" and "Reject"),

decide whether there exists a quantum state $|\psi\rangle$ such that

- YES case: $\sum p_i \leq \alpha$
- NO case: $\sum p_i \geq \beta$

$\alpha < \beta.$

where p_i = $\Pr[\text{measuring}|\psi\rangle \text{ using } H_i \text{ yields "Reject"}]$



Local Hamiltonians problem

(k, α, β)**-Local Hamiltonians problem**: given classical description of measurements $\{H_1, H_2, \ldots, H_m\}$ on n qubits where each H_i

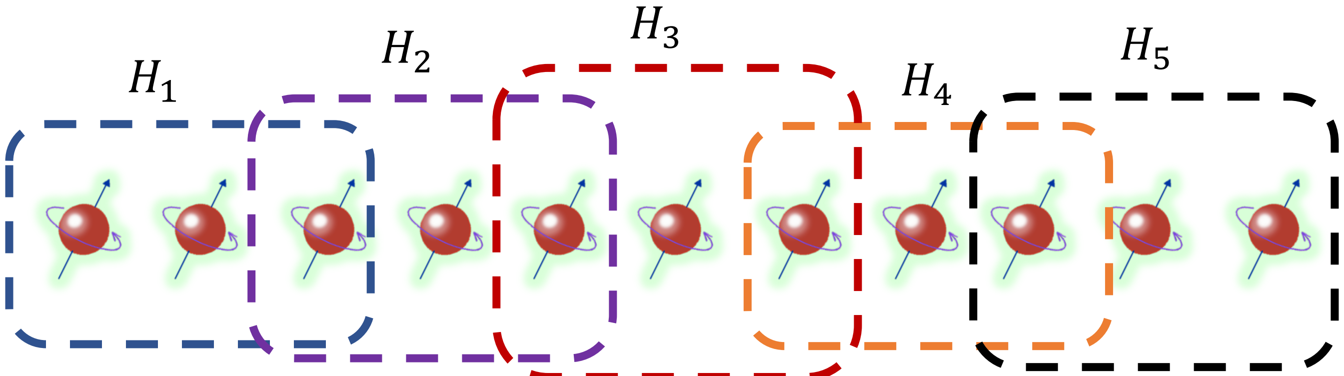
- Acts on k qubits
- Is a two-outcome measurement (with outcomes labelled "Accept" and "Reject"),

decide whether there exists a quantum state $|\psi\rangle$ such that

- <u>YES case:</u> $\sum p_i \leq \alpha$
- <u>NO case:</u> $\sum p_i \geq \beta$

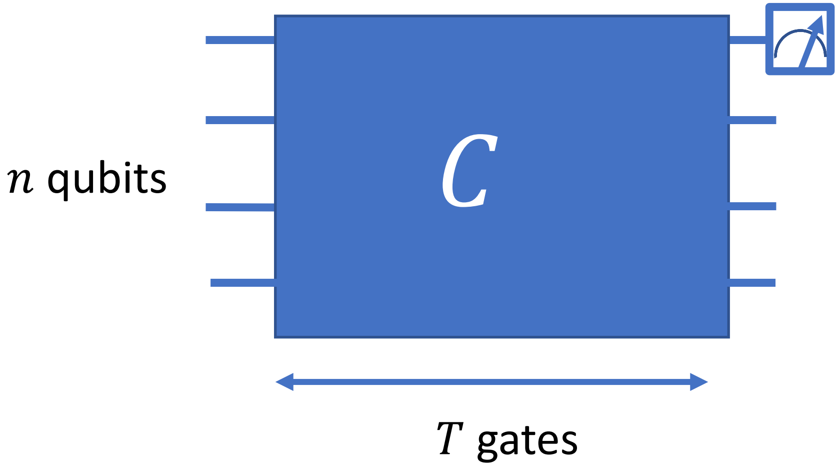
where p_i = $\Pr[\text{measuring}|\psi\rangle \text{ using } H_i \text{ yields "Reject"}]$

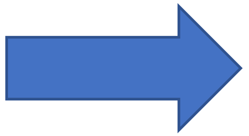
The (k, α, β)-Local Hamiltonians problem is **QMA**-complete for $k = 3, \beta - \alpha \geq \frac{1}{poly(n)}$



QMA-completeness of Local Hamiltonians

Instance of **Q-VER-CIRCUIT**





Instance of **3-Local Hamiltonians**

3-Local Measurements $\{H_1, H_2, \ldots, H_m\}$ where

- <u>YES case:</u> there exists a quantum state $|\psi\rangle$ such that $\sum p_i \leq \exp(-n)$
- <u>NO case:</u> for all quantum states $|\psi\rangle$, $\sum p_i \geq \Omega\left(\frac{1}{T^3}\right)$

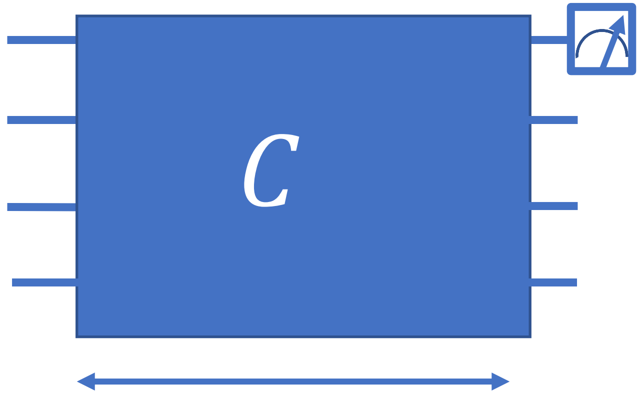
where p_i = $\Pr[\text{measuring}|\psi\rangle \text{ using } H_i \text{ yields "Reject"}]$

Assume that WLOG completeness and soundness errors are exponentially small.

QMA-completeness of Local Hamiltonians

Instance of **Q-VER-CIRCUIT**

n qubits



T gates

Let $|\theta\rangle$ be such that C accepts $|\theta\rangle \otimes |0 \cdots 0\rangle$ with probability at least $1 - \exp(-n)$.

Witnesses of YES instances of **Q-VER-CIRCUIT** are mapped to witnesses of YES instances of **3-Local Hamiltonians** in the following way:

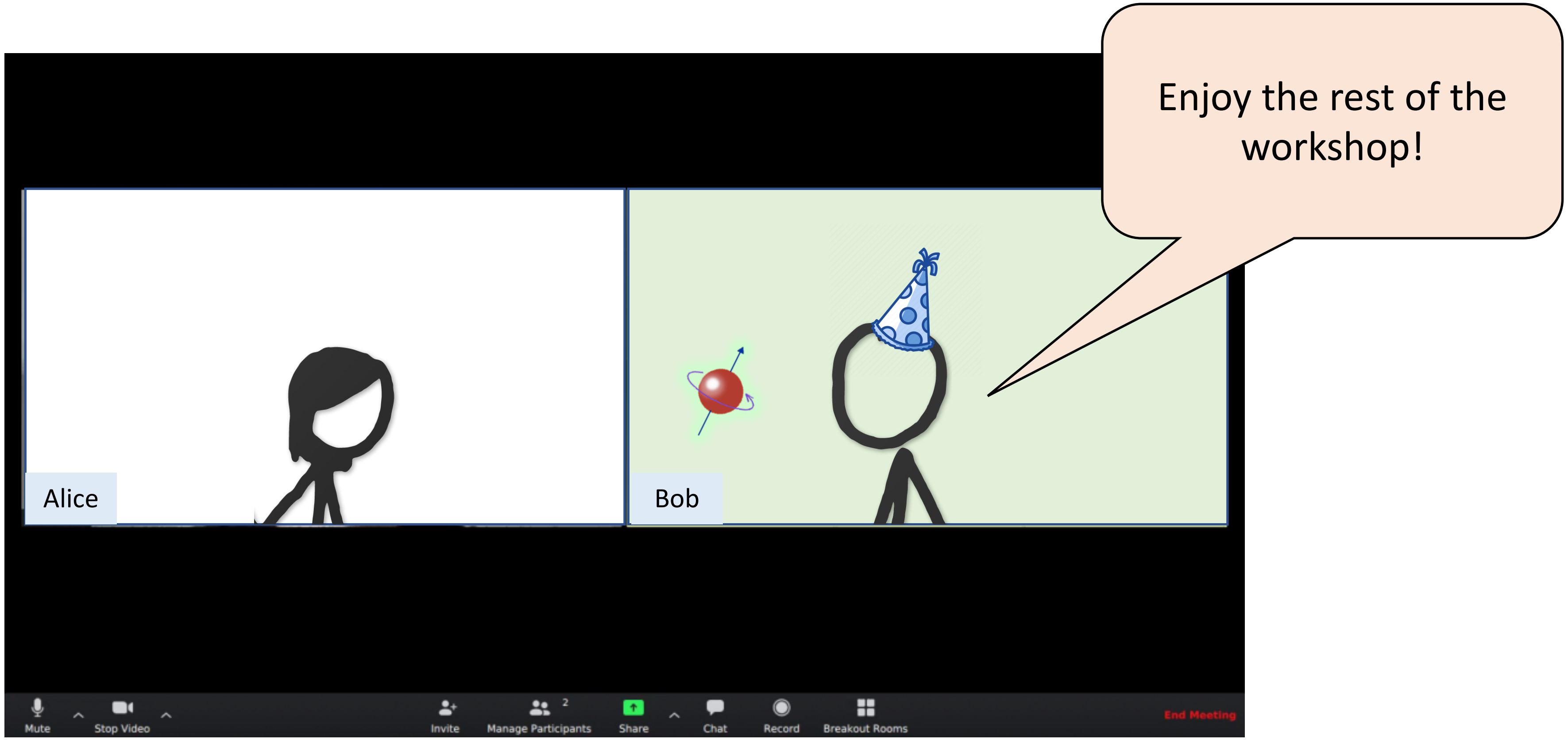
$$|\psi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |\hat{t}\rangle \otimes |\psi_t\rangle$$

"history state"

where

- $|\psi_0\rangle = |\theta\rangle \otimes |0 \cdots 0\rangle$
- $|\psi_t\rangle = G_t |\psi_{t-1}\rangle$ for $t \geq 1$

$$\sum \Pr[\text{measuring} |\psi\rangle \text{ using } H_i \text{ yields "Reject"}] \leq \exp(-n)$$


Enjoy the rest of the workshop!
Alice
Bob
Mute
Stop Video
Invite
Manage Participants
Share
Chat
Record
Breakout Rooms
End Meeting

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