Winter School on Lattice-Based Cryptography and Applications
Bar-Ilan University, Israel 22/2/2012

Bar-Ilan University
Dept. of Computer Science

Proving Hardness of LWE

(based on [R05, J. of the ACM])

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Outline

- Introduction to lattices
- Main theorem: hardness of LWE
- Proof of main theorem
  - Overview
  - Part I: Quantum
  - Part II: Classical

Lattices

Basis:

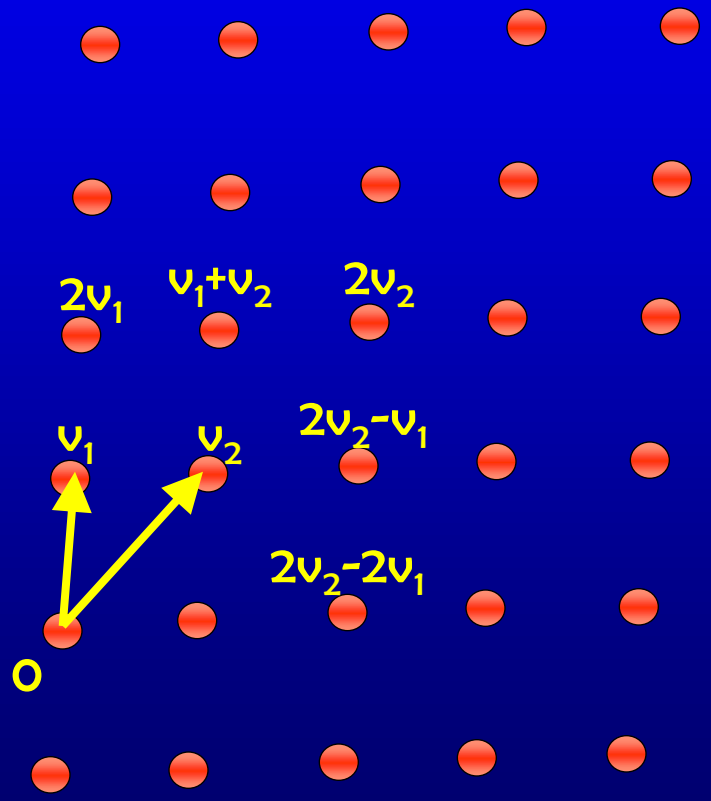
$v_1,\ldots,v_n$ vectors in R^n

The lattice L is

$L=\{a_1v_1+\ldots+a_nv_n \mid a_i \text{ integers}\}$

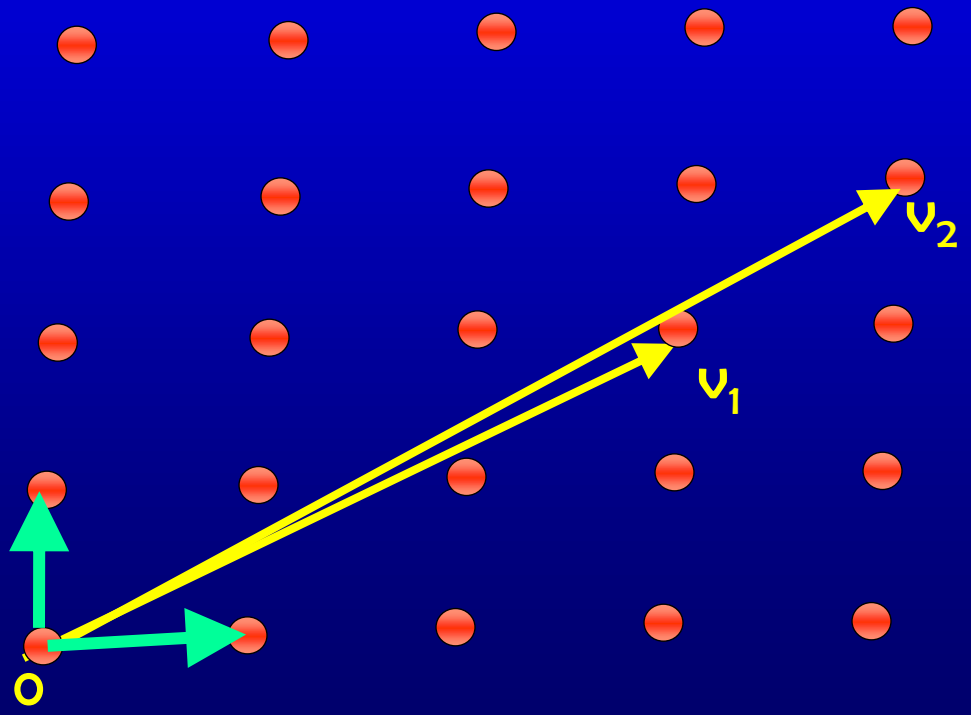
The dual lattice of L is

$L^*=\{x \mid \forall\, y\in L,\ \langle x,y\rangle \in Z\}$



Shortest Independent Vectors Problem (SIVP)

- SIVP: Given a lattice, find a 'short' set of n linearly independent lattice vectors (say within factor n of shortest)



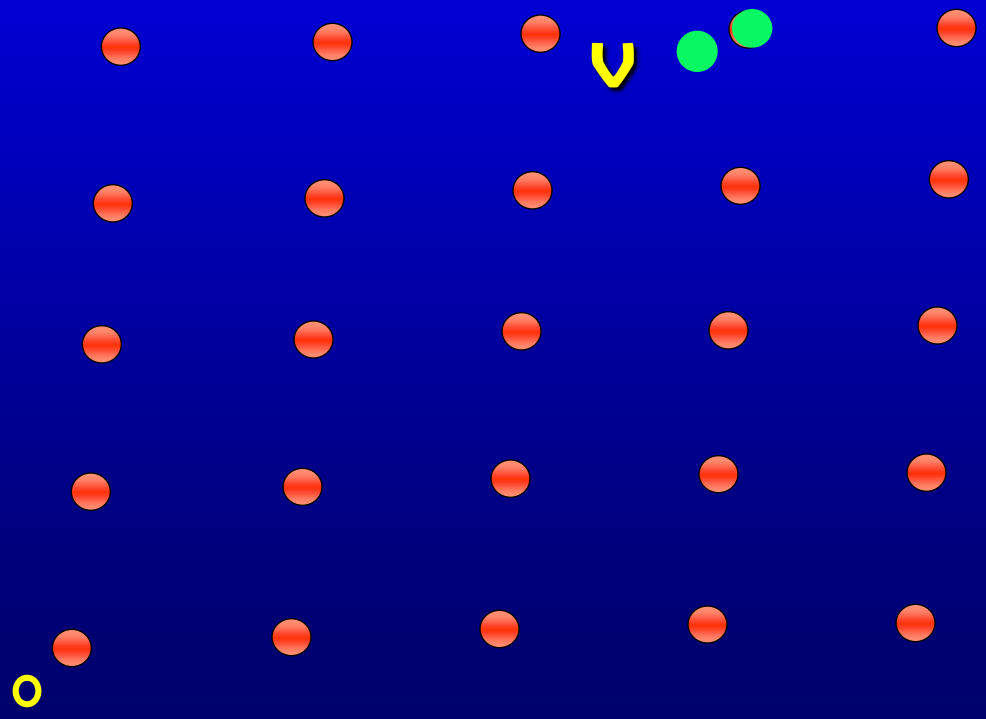
SIVP Seems Hard

- Best known algorithm runs in time 2^n
  [AjtaiKumarSivakumar01,...]
- No better quantum algorithm known!

- On the other hand, not believed to be NP-hard
  [GoldreichGoldwasser00, AharonovR04]

Bounded Distance Decoding

- BDD_d: Given a lattice and a target vector within distance d, find the closest lattice point



Main Theorem

Hardness of LWE

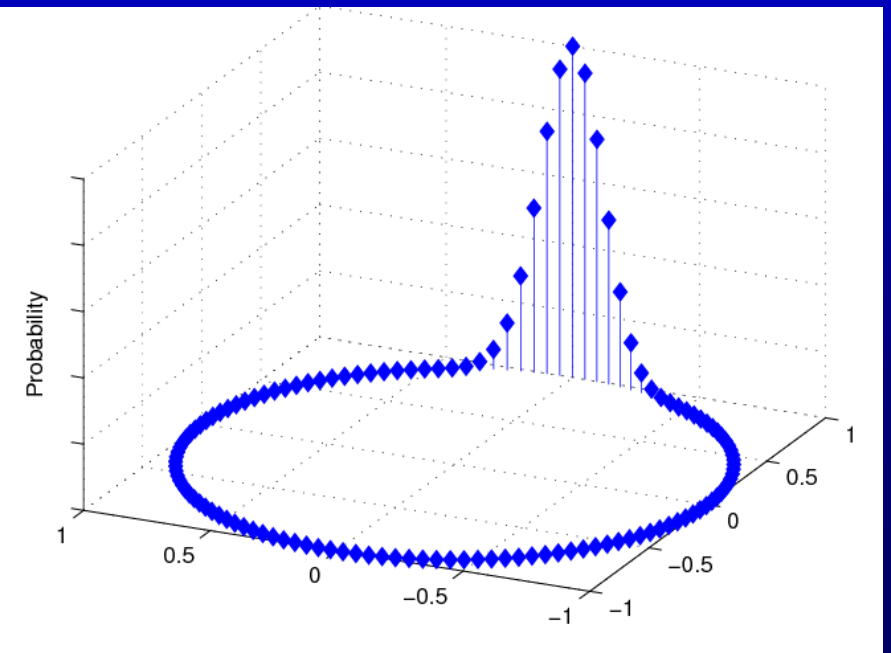
LWE

- Fix some $p < \mathrm{poly}(n)$
- Let $s \in Z_p^{\,n}$ be a secret
- We have random equations modulo p with error:

$$
\begin{aligned}
2s_1+0s_2+2s_3+1s_4+2s_5+4s_6+\ldots+4s_n &\approx 2\\
0s_1+1s_2+5s_3+0s_4+6s_5+6s_6+\ldots+2s_n &\approx 4\\
6s_1+5s_2+2s_3+0s_4+5s_5+2s_6+\ldots+0s_n &\approx 2\\
6s_1+4s_2+4s_3+4s_4+3s_5+3s_6+\ldots+1s_n &\approx 5\\
&\vdots
\end{aligned}
$$

LWE

- More formally, we need to learn s from samples of the form $(t, st+e)$ where t is chosen uniformly from Z_p^n and e is chosen from Z_p
- Easy algorithms need $2^{O(n \log n)}$ equations/time
- Best algorithm needs $2^{O(n)}$ equations/time [BlumKalaiWasserman'00]
- Subexponential algorithm if noise $< \sqrt{n}$ [AroraGe'11]



Main Theorem

LWE is as hard as worst-case lattice problems using a quantum reduction

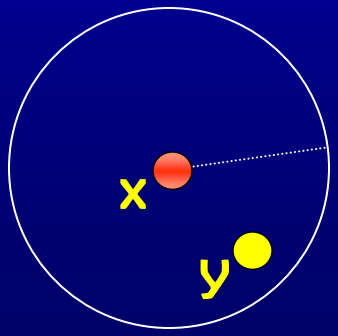
- In other words: solving LWE implies an efficient quantum algorithm for lattices

Why Quantum?

- As part of the reduction, we need to perform a certain algorithmic task on lattices
- We do not know how to do it classically, only quantumly!

Why Quantum?

- We are given an oracle that solves BDD_d for some small d
- As far as I can see, the only way to generate inputs to this oracle is:
  - Somehow choose $x \in L$
  - Let y be some random vector within dist d of x
  - Call the oracle with y
- The answer is x. But we already know the answer !!
- Quantumly, being able to compute x from y is very useful: it allows us to transform the state $|y,x\rangle$ to the state $|y,0\rangle$ reversibly (and then we can apply the quantum Fourier transform)



Proof of the Main Theorem

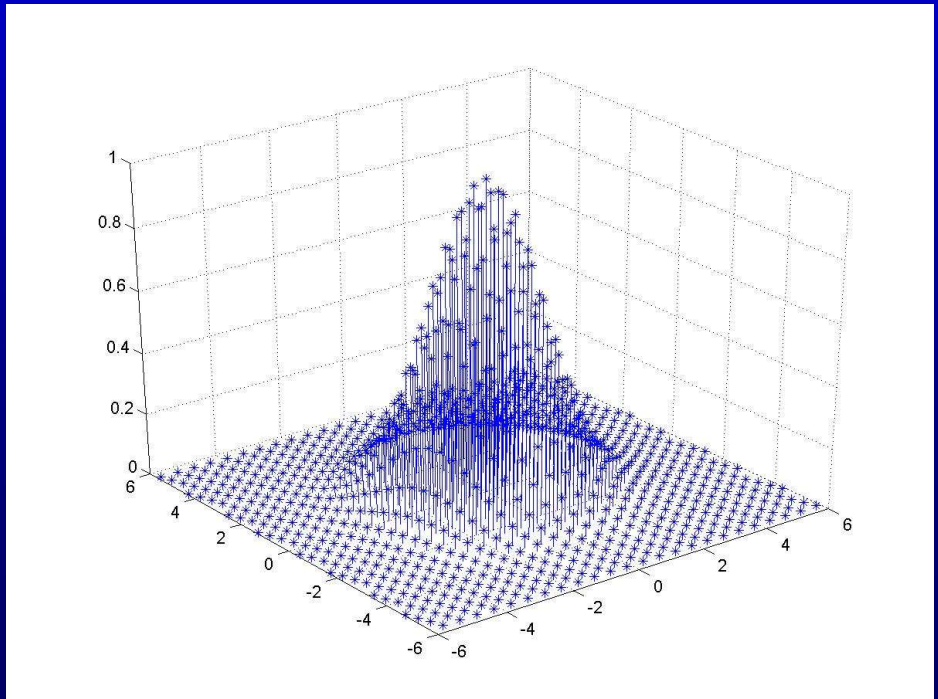
Overview

Gaussian Distribution

- Recall the discrete Gaussian distribution on a lattice (normalization omitted):

$$\forall x \in L,\ D_r(x) = e^{-\|x/r\|^2}$$

- We can efficiently sample from D_r for large $r=2^n$

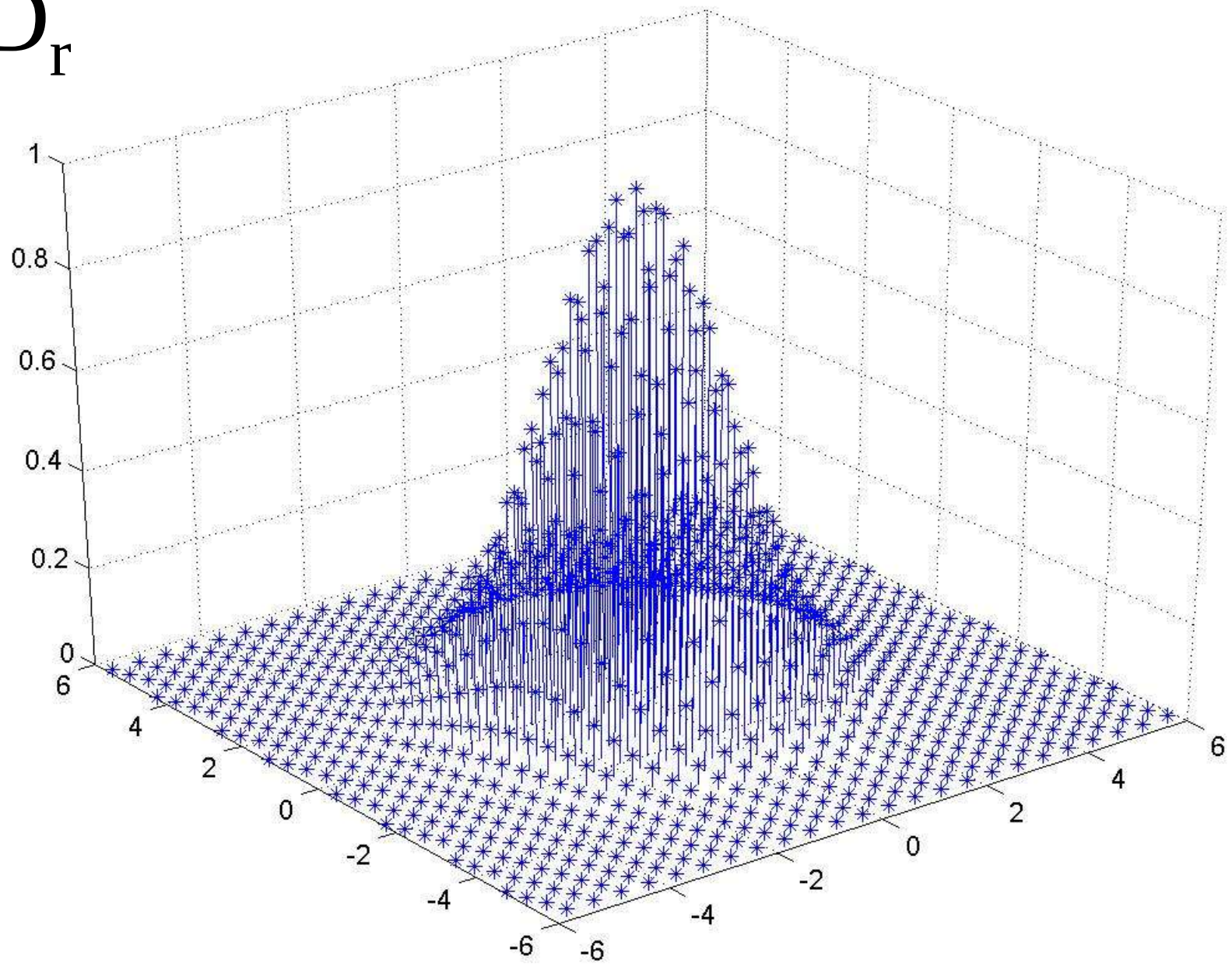


The Reduction

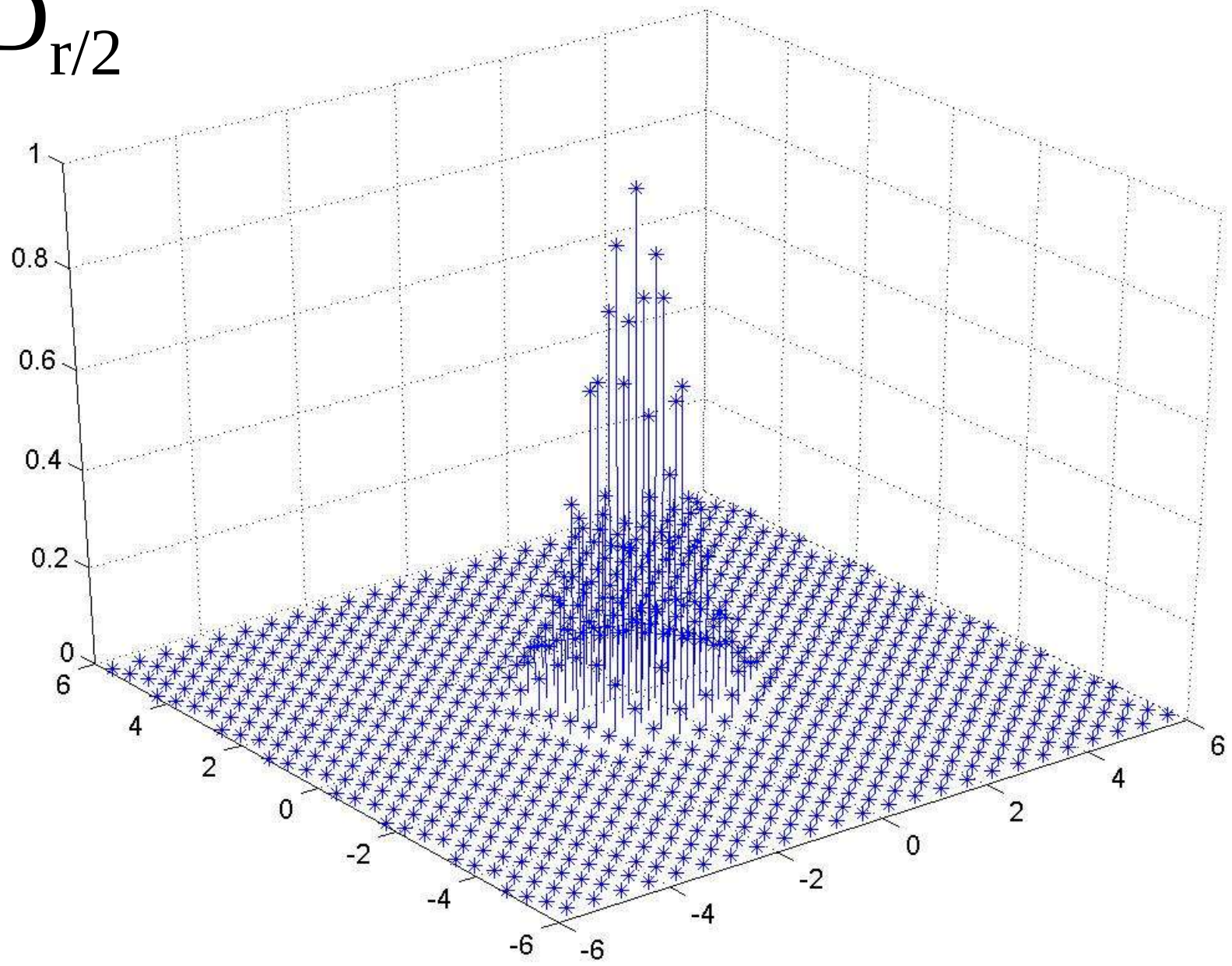
- Assume the existence of an algorithm for LWE for $p=2\sqrt{n}$

- Our lattice algorithm:
  - $r=2^n$
  - Take poly(n) samples from D_r
  - Repeat:
    - Given poly(n) samples from D_r compute poly(n) samples from $D_{r/2}$
    - Set $r \leftarrow r/2$
  - When r is small, output a short vector

D_r



$D_{r/2}$



Obtaining $D_{r/2}$ from D_r

$p=2\sqrt{n}$

- Lemma 1:

  Given poly(n) samples from D_r, and an LWE oracle, we can solve $BDD_{p/r}$ in L^*

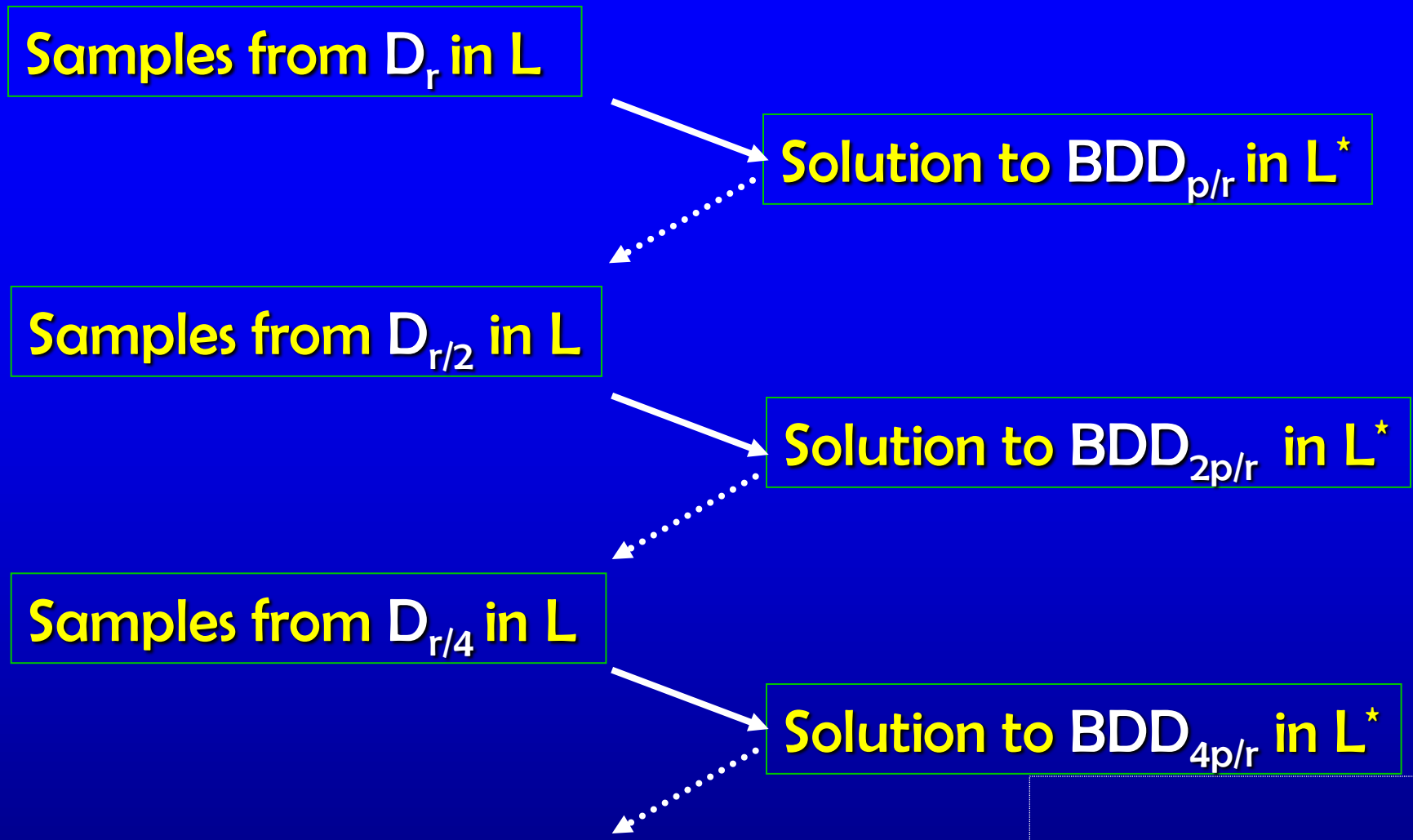
  - Classical

- Lemma 2:

  Given a solution to BDD_d in L^*, we can obtain samples from $D_{\sqrt{n}/d}$

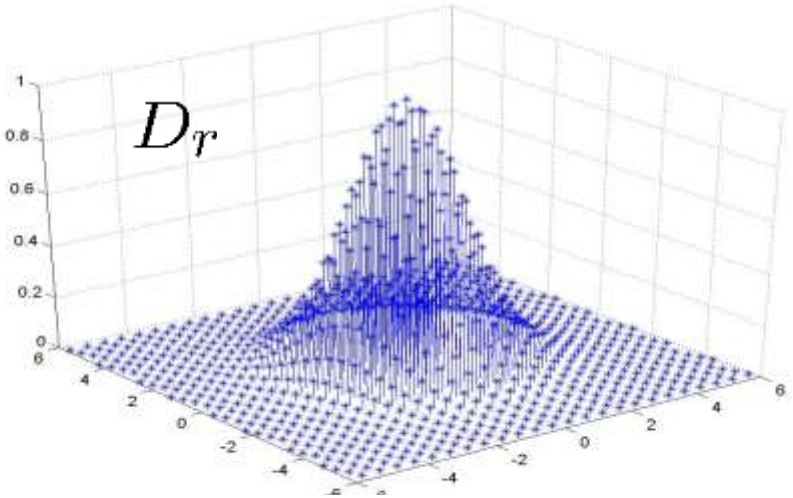
  - Quantum
  - Based on the quantum

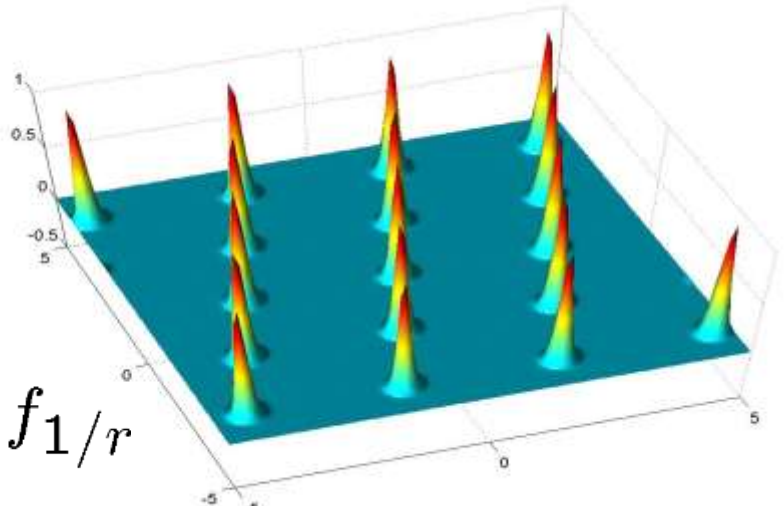
    Fourier transform



Classical, uses LWE oracle

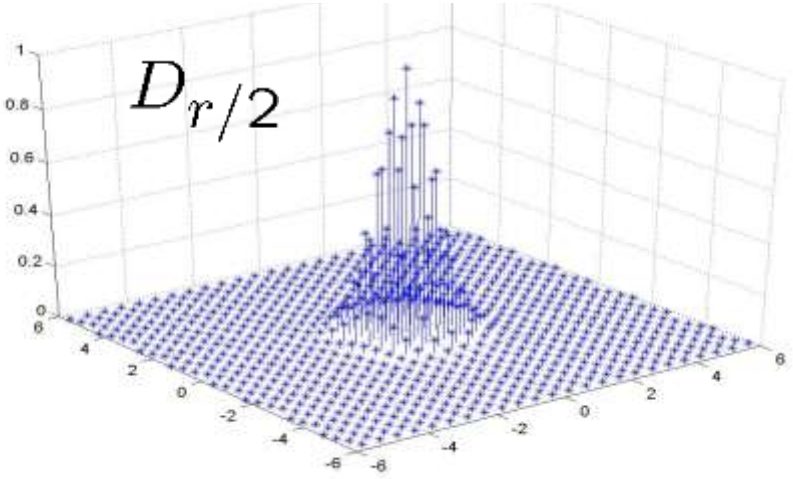
Quantum

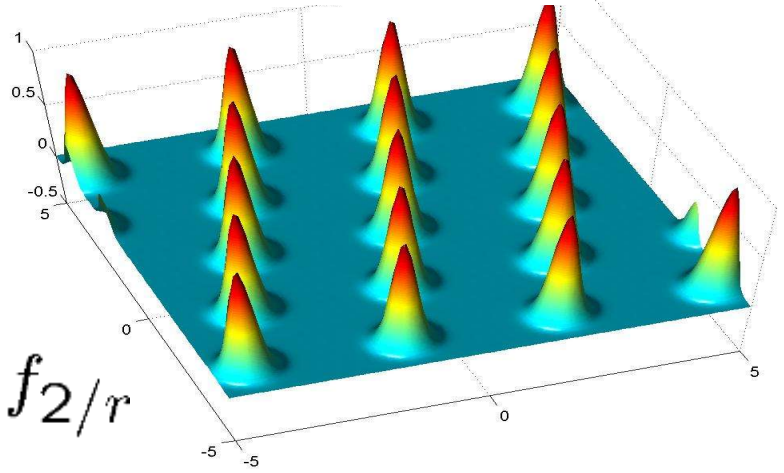




Primal world (L)

Dual world (L*)





Fourier Transform

- The Fourier transform of D_r is given by

$$f_{1/r}(x) \approx e^{-\|r \cdot \mathrm{dist}(x, L^*)\|^2}$$

- Its value is
  - 1 for x in L^*,
  - e^{-1} at points of distance 1/r from L^*,
  - $\approx$0 at points far away from L^*.

Proof of the Main Theorem

Lemma 2: Obtaining $D_{\sqrt{n/d}}$ from BDD_d

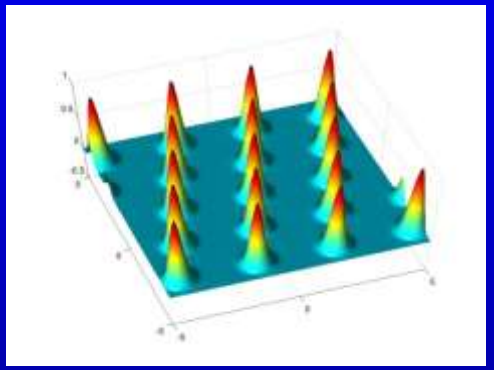
From BDD_d to $D_{\sqrt{n}/d}$

- Assume we can solve BDD_d; we'll show how to obtain samples from $D_{\sqrt{n}/d}$

- <u>Step 1:</u>

  Create the quantum state

$$\sum_{x \in \mathbb{R}^n} f_{d/\sqrt{n}}(x)|x\rangle$$



by adding a Gaussian to each lattice point and uncomputing the lattice point using the BDD algorithm

From BDD_d to $D_{\sqrt{n}/d}$

- Step 2:

  Compute the quantum Fourier transform of

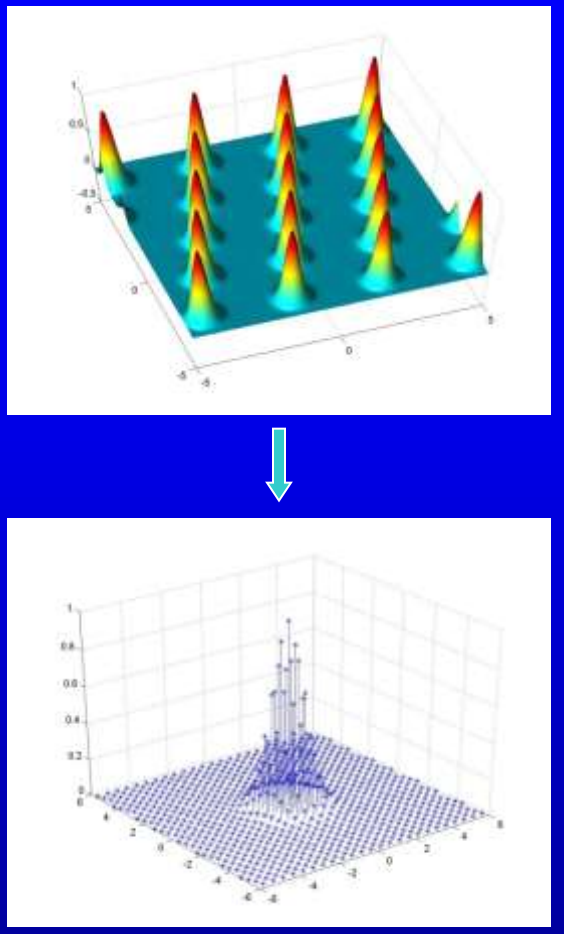
$$\sum_{x \in \mathbb{R}^n} f_{d/\sqrt{n}}(x)|x\rangle$$

  It is exactly $D_{\sqrt{n}/d}$!!

- Step 3:

  Measure and obtain one sample from $D_{\sqrt{n}/d}$

- By repeating this process, we can obtain poly(n) samples



Proof of the Main Theorem

Lemma 1: Solving $BDD_{p/r}$ given samples from D_r and an LWE oracle

It's enough to approximate $f_{p/r}$

- Lemma: being able to approximate $f_{p/r}$ implies a solution to $BDD_{p/r}$
- Proof Idea – walk uphill:
  - $f_{p/r}(x) > 1/4$ for points x of distance < p/r
  - Keep making small modifications to x as long as $f_{p/r}(x)$ increases
  - Stop when $f_{p/r}(x)=1$ (then we are on a lattice point)

What's ahead in this part

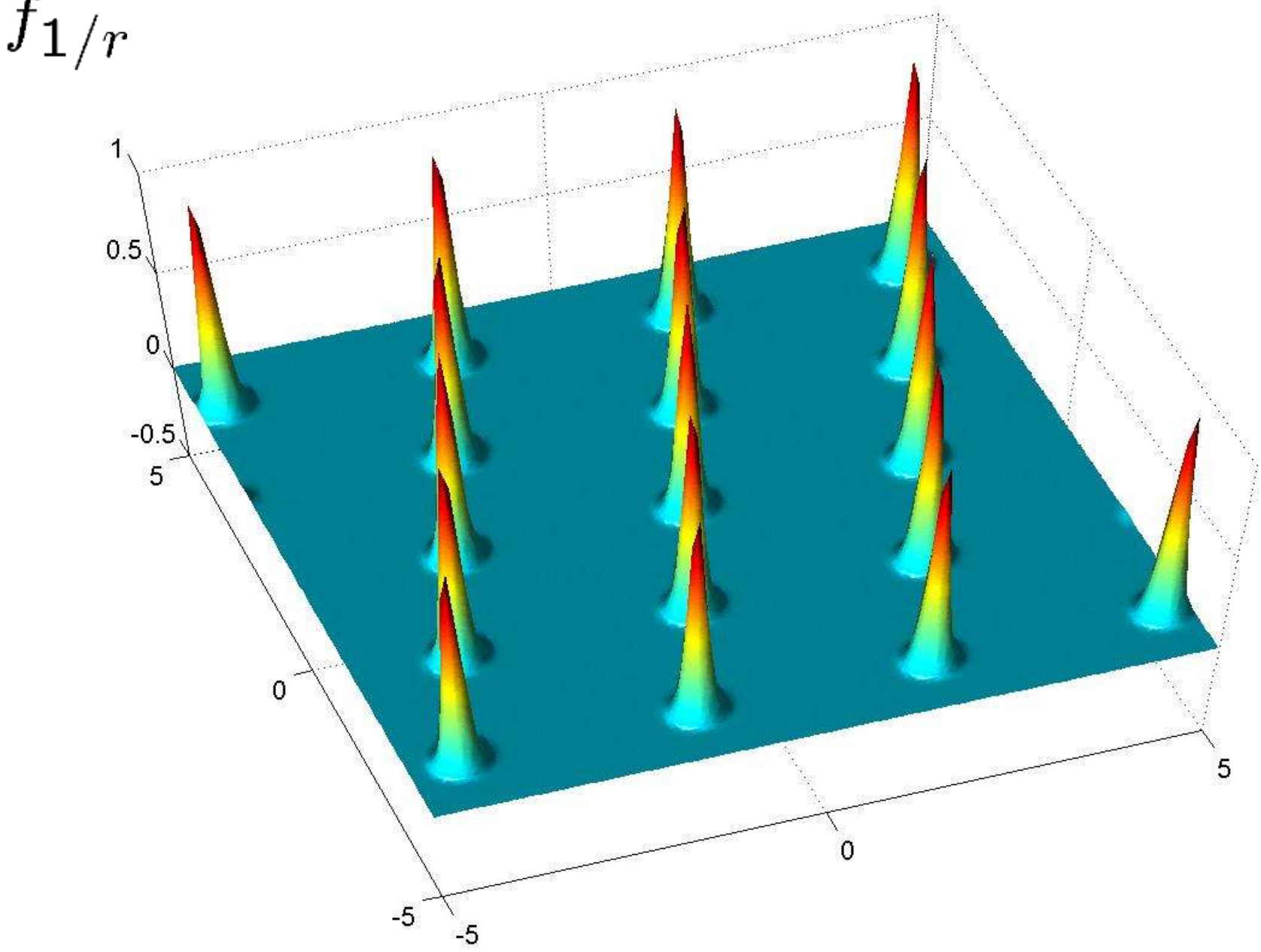
- For warm-up, we show how to approximate $f_{1/r}$ given samples from D_r
  - No need for the LWE oracle
  - This is main idea in [AharonovR'04]
- Then we show how to approximate $f_{2/r}$ given samples from D_r and an LWE oracle (for p=2)
- Approximating $f_{p/r}$ is similar

Warm-up: approximating $f_{1/r}$

- Let's write $f_{1/r}$ in its Fourier representation:

$$f_{1/r}(x) = \sum_{w \in L} \widehat{f_{1/r}}(w) \cos(2\pi\langle w, x\rangle)$$
$$= \sum_{w \in L} D_r(w) \cos(2\pi\langle w, x\rangle)$$
$$= E_{w \sim D_r} [\cos(2\pi\langle w, x\rangle)]$$

- Using samples from D_r, we can compute a good approximation to $f_{1/r}$ (this is the main idea in [AharonovR'04])


$f_{1/r}$
1
0.5
0
-0.5
5
0
-5
-5
0
5

Fourier Transform

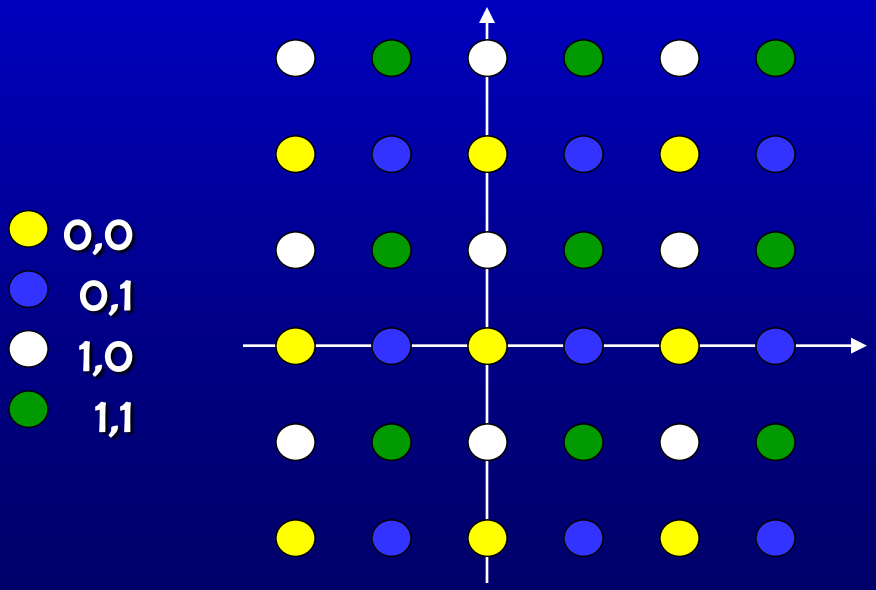
- Consider the Fourier representation again:

$$f_{1/r}(x) = E_{w \sim D_r}\left[\cos(2\pi\langle w, x\rangle)\right]$$

- For $x \in L^*$, $\langle w,x \rangle$ is integer for all w in L and therefore we get $f_{1/r}(x)=1$
- For x that is close to L^*, $\langle w,x \rangle$ is distributed around an integer. Its standard deviation can be (say) 1.

Approximating $f_{2/r}$

- Main idea: partition D_r into 2^n distributions
- For $t \in (Z_2)^n$, denote the translate t by D^t_r
- Given a lattice point we can compute its t
- The probability on $(Z_2)^n$ obtained by sampling from D_r and outputting t is close to uniform



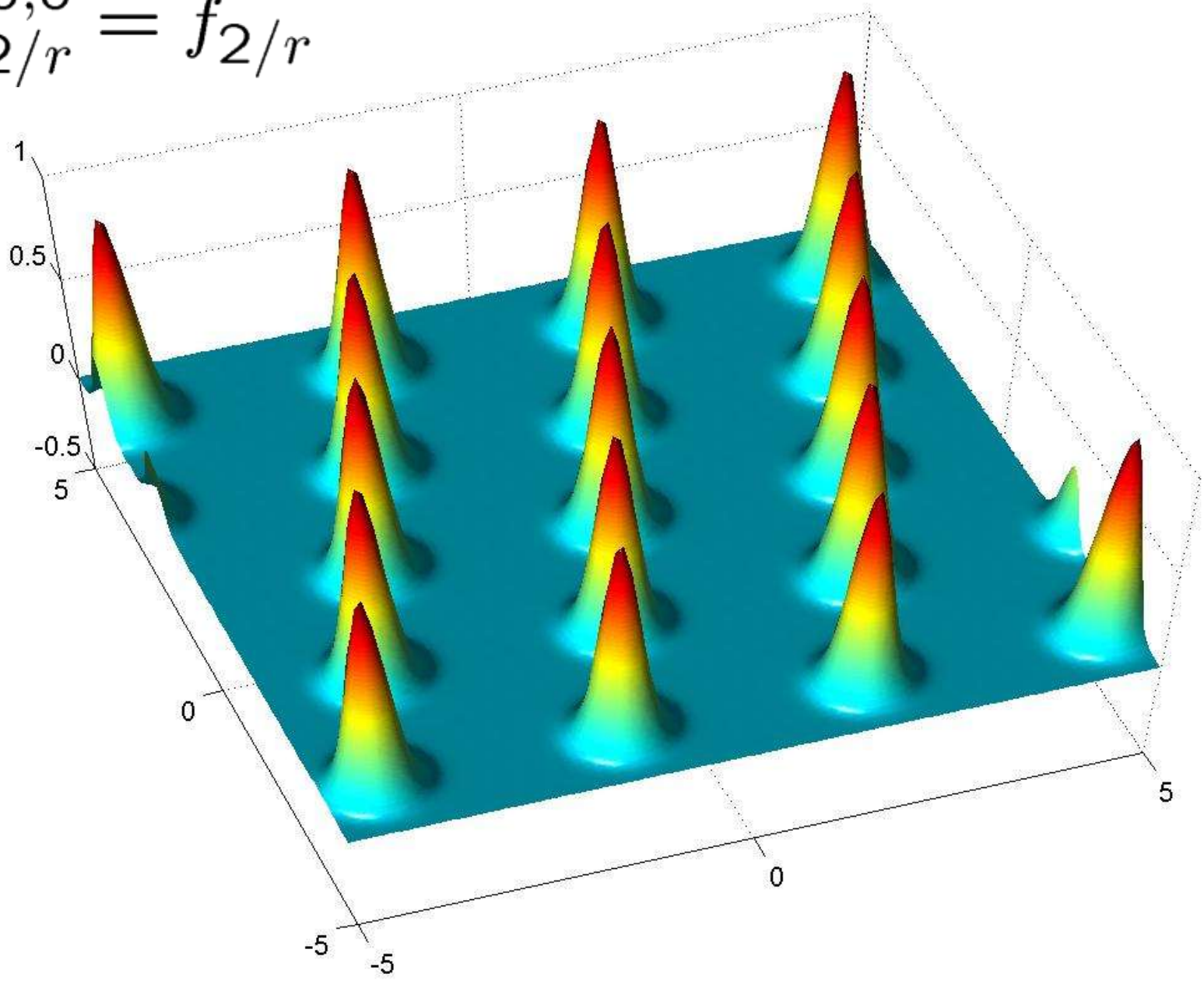
Approximating $f_{2/r}$

- Hence, by using samples from D_r we can produce samples from the following distribution on pairs (t,w):
  - Sample $t \in (Z_2)^n$ uniformly at random
  - Sample w from D^t_r

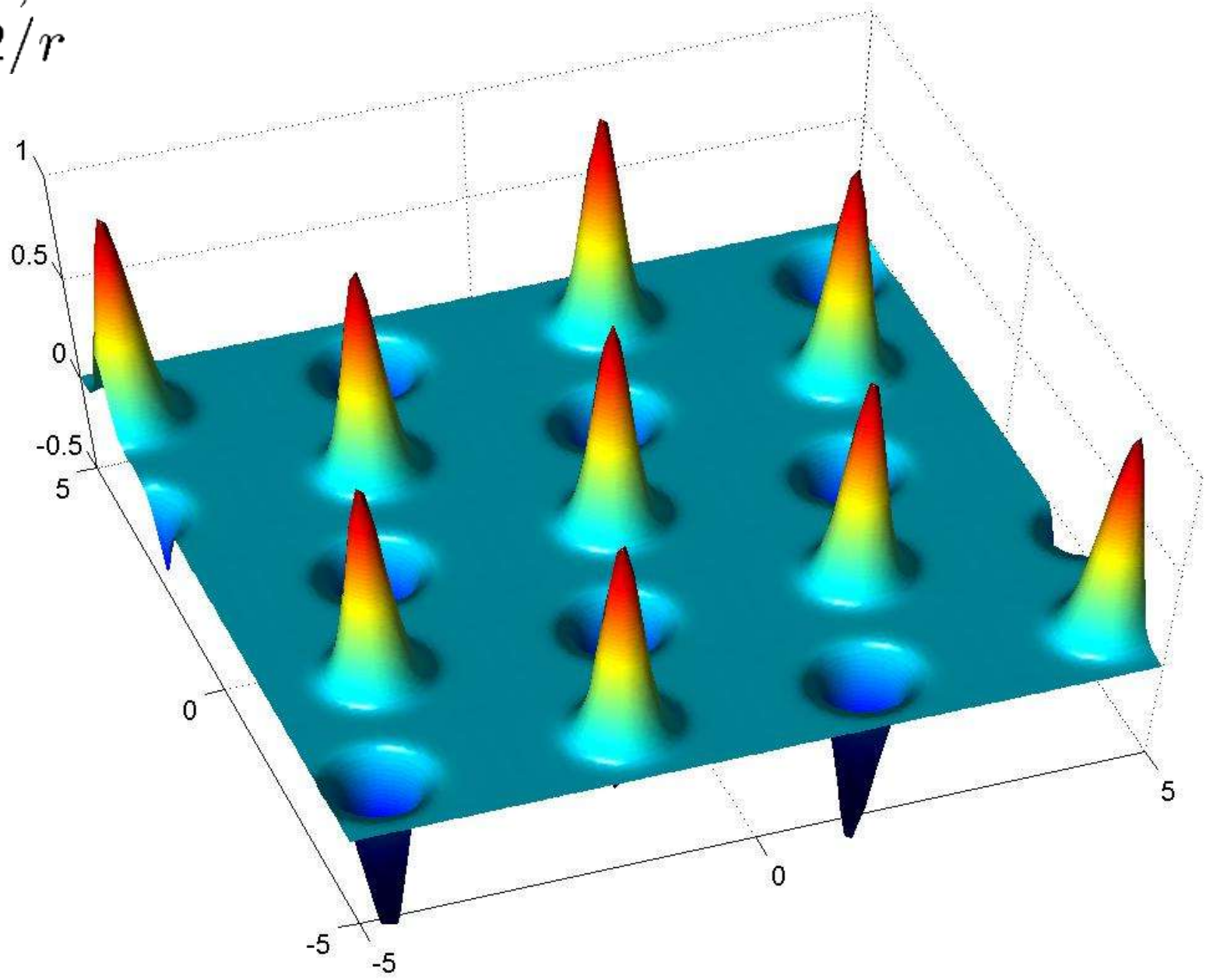
- Consider the Fourier transform of D^t_r

$$f^t_{2/r}(x) = E_{w \sim D^t_r}[\cos(\pi \langle w, x \rangle)]$$

$$f^{0,0}_{2/r} = f_{2/r}$$



$f_{2/r}^{1,1}$



Approximating $f_{2/r}$

- The functions $f^t_{2/r}$ look almost like $f_{2/r}$
- Only difference is that some Gaussians have their sign flipped
- Approximating $f^t_{2/r}$ is enough: we can easily take the absolute value and obtain $f_{2/r}$
- For this, however, we need to obtain several pairs (t,w) for the same t

- The problem is that each sample (t,w) has a different t !

Approximating $f_{2/r}$

- Fix x close to L^*
- The sign of its Gaussian is ± 1 depending on $\langle s,t \rangle \mod 2$ for $s \in (Z_2)^n$ that depends only on x
- The distribution of $\langle x,w \rangle \mod 2$ when w is sampled from D^t_r is centred around $\langle s,t \rangle \mod 2$
- Hence, we obtain equations modulo 2 with error:

$$\begin{aligned} \langle s,t_1 \rangle &\approx \lceil \langle x,w_1 \rangle \rfloor \mod 2 \\ \langle s,t_2 \rangle &\approx \lceil \langle x,w_2 \rangle \rfloor \mod 2 \\ \langle s,t_3 \rangle &\approx \lceil \langle x,w_3 \rangle \rfloor \mod 2 \\ &\vdots \end{aligned}$$

Approximating $f_{2/r}$

- Using the LWE oracle, we solve these equations and obtain s
- Knowing s, we can cancel the sign
- Averaging over enough samples gives us an approximation to $f_{2/r}$

Open Problems

1. What happens for small moduli, say p=2 (learning parity with noise (LPN))?
2. Dequantize the reduction:
   - This would immediately improve the security of all LWE-based crypto
   - Main obstacle: what can one do classically with a solution to BDD_d? (see [Peikert09])
3. Use quantum hardness assumptions to prove security of other cryptosystems

More Recent Work

- [Peikert09] classical reduction, but exponential modulus and based on GapSVP only
- [StehléSteinfeldTanakaXagawa09] direct quantum reduction from SIS to LWE using the quantum part (but gives weaker hardness of LWE), as well as a ring version of LWE
- [LyubashevskyPeikertR09] Ring-LWE

Thanks !!