Winter School on Lattice-Based Cryptography and Applications
Bar-Ilan University, Israel 19/2/2012-22/2/2012

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Fully Homomorphic Encryption

Craig Gentry
IBM Watson

Outline for Today

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- Homomorphic Encryption Basics
- Somewhat homomorphic encryption (SWHE) schemes

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Homomorphic Encryption Basics

Homomorphic Encryption Basics

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A way to delegate processing of your data, without giving away access to it.

Example App: Cloud computing on encrypted data

Do you really think it's safe to store your data in the cloud *unencrypted*?

"Where the sensitive information is concentrated, that is where the spies will go. This is just a fact of life." – Ken Silva, former NSA official

Fully Homomorphic Encryption

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The special sauce!

Run Evaluate[f, $Enc_k(x)$] = $Enc_k[f(x)]$

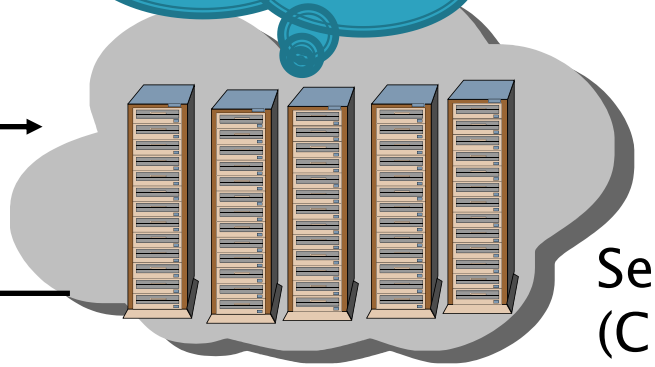
"I want 1) the cloud to process my data
2) even though it is encrypted.

Alice
(Input: data x, key k)

$Enc_k(x)$

function f

This could be encrypted too.



Server (Cloud)

$Enc_k[f(x)]$

f(x)

An Analogy: Alice's Jewelry Store

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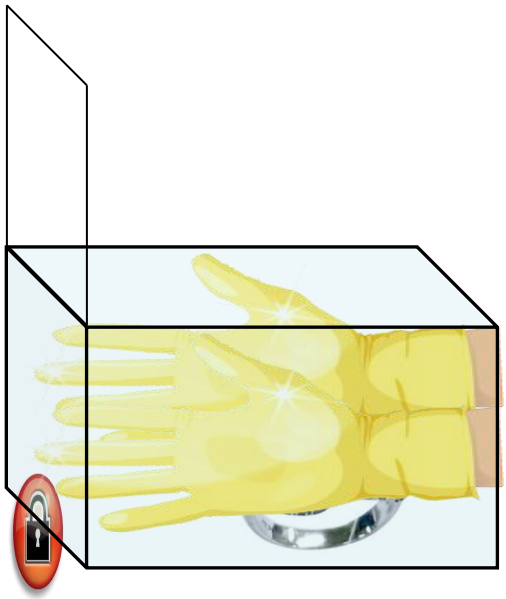
- Alice wants workers to assemble raw materials into jewelry
- But Alice is worried about theft:

  She wants her workers to process the raw materials without having access to them.



- Alice puts raw materials in locked glovebox.
- Workers assemble jewelry inside glovebox, using the gloves.
- Alice unlocks box to get "results".

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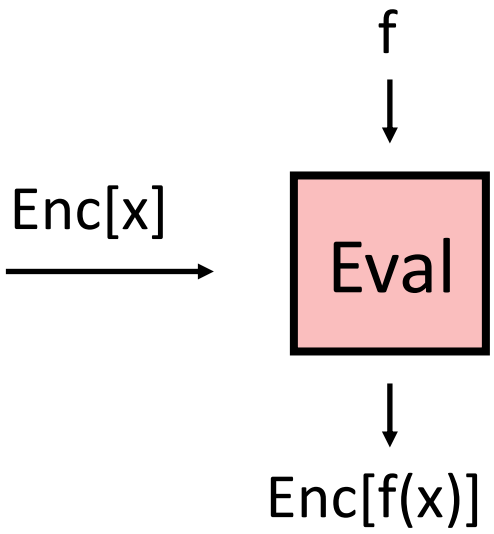
Homomorphic Encryption [RAD78]:



Compactness: Size of Eval'd ciphertext independent of f

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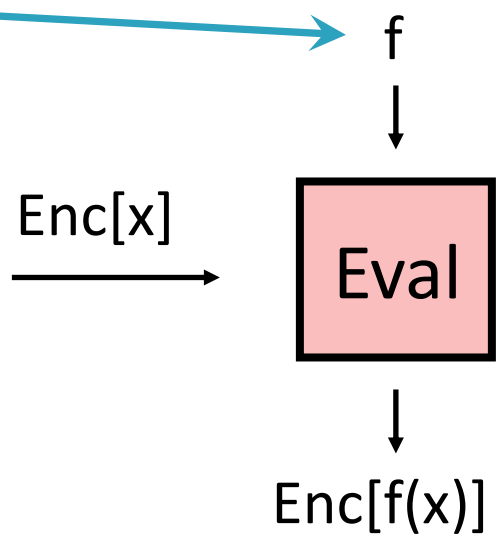
"Fully" means it works for all functions f

Fully Homomorphic Encryption (FHE) [RAD78, Gen09]:



Compactness: Size of Eval'd ciphertext independent of f

Homomorphic Encryption Basics

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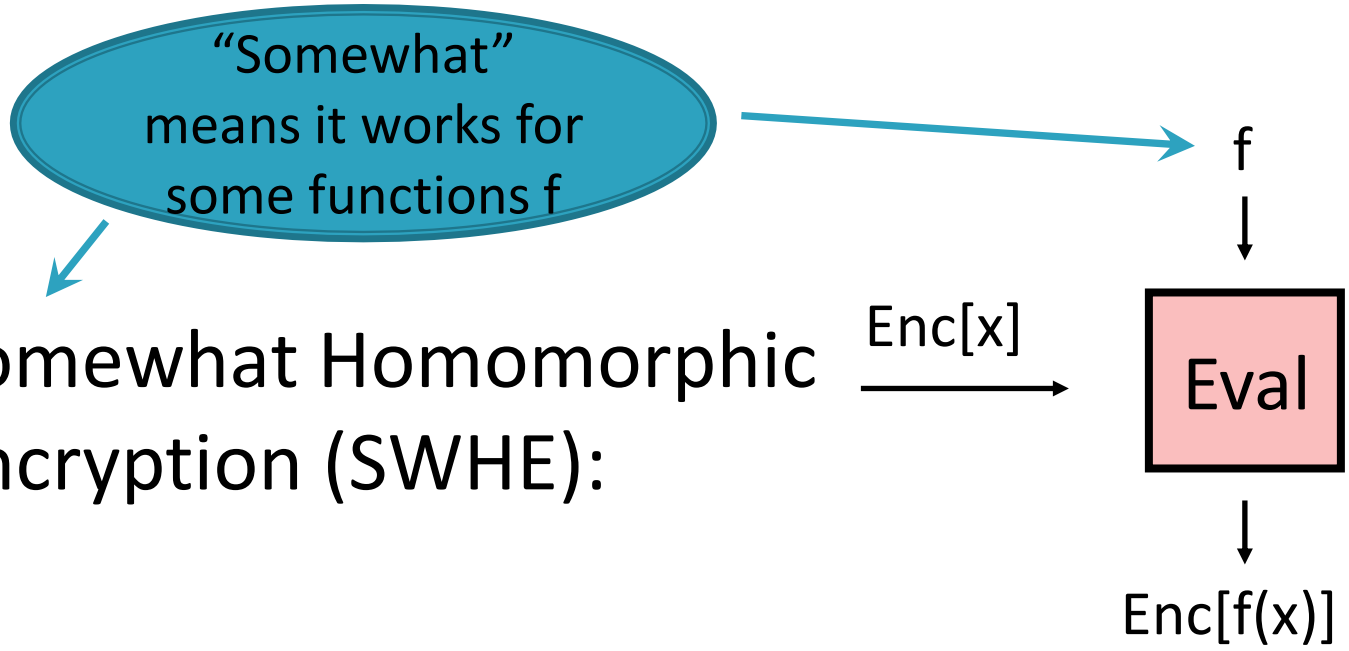

Compactness: Size of Eval'd ciphertext independent of f

Homomorphic Encryption Basics

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A way to delegate processing of your data, without giving away access to it.

- Fully Homomorphic Encryption (FHE):
  - Arbitrary processing
  - But computationally expensive.
- Somewhat Homomorphic Encryption (SWHE):
  - Limited processing
  - Cheaper computationally.

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Homomorphic Encryption Basics: Functionality

Processing (Unencrypted) Data

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- Forget encryption for a moment…
- How does your computer compute a function?
- Basically, by working on *bits*, 1's and 0's.
- And by using bit operations – for example,
  - $\mathrm{AND}(b_1, b_2) = 1$ if $b_1 = b_2 = 1$; otherwise, equals 0.
    - $\mathrm{AND}(b_1, b_2) = b_1 \times b_2$.
  - $\mathrm{XOR}(b_1, b_2) = 0$ if $b_1 = b_2$; equals 1 if $b_1 \neq b_2$.
    - $\mathrm{XOR}(b_1, b_2) = b_1 + b_2$ (modulo 2)
- *Any* function can be computed bit-wise – with only ANDs and XORs – if it can be computed at all.

Unencrypted String Matching

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- Still forget encryption for now…
- Example: How do you detect whether a string is in a file?

Step 1: Match string against subsequences of file

```
01100111101100100100010001
  XOR
111011
------
100010
```

`ZeroString(100010) = 0`
(not the zero string! not a match!)

The `ZeroString` function itself can be computed from basic bit operations.

Unencrypted String Matching

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- Still forget encryption for now…
- Example: How do you detect whether a string is in a file?

Step 1: Match string against subsequences of file

```
01100111101100100100010001
XOR
111011
------
000000
```

`ZeroString(000000) = 0`
(is the zero string! a match!)

Unencrypted String Matching

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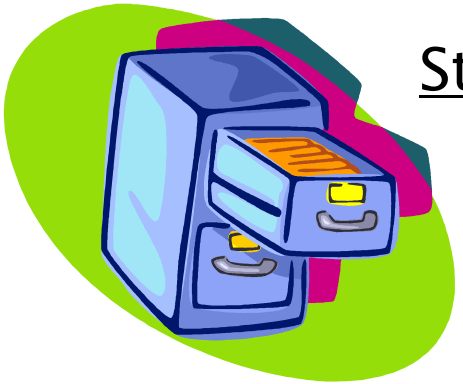
- Still forget encryption for now…
- Example: How do you detect whether a string is in a file?



Step 2: Aggregate info about the subsequences

0110011110110010010010001

1110000000000011

↓↓↓↓↓↓↓↓

00000010…

OR(00000010…) = 1 (string is in the file!)

OR also can be decomposed into ANDs and XORs.

Let's Do This Encrypted…

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- Let $\boxed{b}$ denote a valid encryption of bit b.
- Suppose we have a (homomorphic) encryption scheme with public functions E-ADD, E-MULT where:

$$\text{E-MULT}(\boxed{b_1},\boxed{b_2}) = \boxed{b_1 \times b_2} \quad \text{E-ADD}(\boxed{b_1},\boxed{b_2}) = \boxed{b_1 + b_2}$$

  for any $\boxed{b_1}$ and $\boxed{b_2}$.
- Then we can AND and XOR *encrypted* bits.
- Proceeding bit-wise, we can compute *any* function on *encrypted* data.

Encrypted String Matching

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$\boxed{b}$ denotes an encryption of bit b.

Bit-wise encrypted file

Step 1: Match string against subsequences of file

$\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}$

E-ADD

$\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{1}$

$\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}$

E-ZeroString($\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}$) = $\boxed{0}$

(not the zero string! not a match!)

E-ZeroString function itself can be computed from basic bit operations.

Encrypted String Matching

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$\boxed{b}$ denotes an encryption of bit b.

Bit-wise encrypted file

Step 2: Aggregate info about the subsequences

$\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{1}$

$\boxed{1}\boxed{1}\boxed{1}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{1}$

↓ ↓ ↓ ↓ ↓ ↓ ↓ ↓

$\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}$

E-OR($\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{0}\boxed{1}\boxed{0}$..) = 1
(string is in the encrypted file!)

E-OR can also be computed from basic bit operations.

Computing General Functions

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- Can you add and multiply (mod 2) and remember stuff?
  - Congratulations, then you can compute any efficiently computable function.
  - If you only can add and multiply mod 3, no worries.
- {ADD,MULT} are Turing-complete (over any ring).
  - Take any (classically) efficiently computable function. Express it as a poly-size circuit of ADD and MULT gates.
- Circuits vs. Turing machines (about the same):
  - Circuit size = $O(T_f \log T_f)$

    T_f = time to compute f on a TM

FHE Defined

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Can your cryptosystem encrypt 0 and 1, and ADD and MULT encrypted data efficiently?

Functionality: Let S_{sk} be set of "valid" ciphertexts for (any) sk.

For $c_1, c_2 \in S_{sk}$, set $c_{ADD} = ADD(c_1, c_2)$, $c_{MULT} = MULT(c_1, c_2)$. Then:

$$DEC_{sk}(c_{ADD}) = DEC_{sk}(c_1) + DEC_{sk}(c_2), \text{ and}$$

$$DEC_{sk}(c_{MULT}) = DEC_{sk}(c_1) \cdot DEC_{sk}(c_2)$$

Also, c_{ADD} and c_{MULT} are in S_{sk}.

In "leveled" FHE, key size may grow with depth of the circuit.

Efficiency: For security parameter k,

- All ops (KEYGEN,ENC,DEC,ADD,MULT) take poly(k) time.
- All valid ciphertexts have poly(k) size.

CPA Security: Best known attacks have complexity 2^k.

Congratulations, you have a (fully) homomorphic encryption scheme!

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Homomorphic Encryption Basics: Security

Security of Homomorphic Encryption

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- Semantic security [GM'84]: For any $m_0 \neq m_1$,
  $(pk, \mathrm{Enc}_{pk}(m_0)) \approx (pk, \mathrm{Enc}_{pk}(m_1))$
  - $\approx$ means indistinguishable by efficient algorithms.
  - pk is a public key, if there is one.
  - Any semantically secure encryption scheme must be probabilistic – i.e., many ciphertexts per plaintext.
- What about IND-CCA1 and IND-CCA2 security?
- IND-CCA2 is impossible for HE, since the adversary can homomorphically tweak the challenge ciphertext.
- IND-CCA1 FHE is open.
- [LMSV10] IND-CCA1 SWHE

Function Privacy

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- Function-privacy: $c^* = \text{Eval}(f, \text{Enc}_{pk}(x))$ hides f.
  - Statistical (when Eval is randomized): c^* has the same distribution as Enc(f(x)).
  - Computational: c^* may not look like a "fresh" ciphertext as long as it decrypts to f(x).

HE Security: A Paradox?

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- Cloud stores my encrypted files: pk, $Enc_{pk}(f_1),\ldots, Enc_{pk}(f_n)$.
- Later, I want f_3, but want to hide "3" from cloud.
- I send $Enc_{pk}(3)$ to the cloud.
- Cloud runs $Eval_{pk}$ (f, $Enc_{pk}(3)$, $Enc_{pk}(f_1),\ldots, Enc_{pk}(f_n)$), where f(n, {files}) is the function that outputs the nth file.
- It sends me the (encrypted) f_3.
- Paradox?: Can't the cloud just "see" it is sending the 3rd encrypted file? By just comparing the stored value $Enc_{pk}(f_3)$ to the ciphertext it sends?

Resolution of paradox:
Semantic security implies:
- Many encryptions of f_3,
- Hard to tell when two ciphertexts encrypt the same thing.

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Homomorphic Encryption Basics: Limitations

FHE Doesn't Do RAM

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- Circuits vs. RAMs:
  - Circuits are powerful: For all functions, circuit-size $\approx$ TM complexity.
  - *But* random-access machines compute some functions much faster than a TM or circuit (Binary search)
  - Can't do "random access" on encrypted data without leaking some information (not surprising)
- What we can do:
  - [GKKMRV11]: "Secure Computation with Sublinear Amortized Work"
  - After setup cost quasi-linear in the size of the data, client and cloud run oblivious RAM on the client's encrypted data.

FHE Doesn't Do Obfuscation

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- Obfuscation:
  - I give the cloud an "encrypted" program E(P).
  - For any input x, cloud can compute E(P)(x) = P(x).
  - Cloud learns "nothing" about P, except $\{x_i, P(x_i)\}$.
- [BGIRSVY01]: "On the (Im)possibility of Obfuscating Programs"
- Difference between obfuscation and FHE:
  - In FHE, cloud computes E(P(x)), and it can't decrypt to get P(x).

FHE Doesn't Do Multi-Key

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- Multi-Key FHE
  - Different clients encrypt data under different FHE keys.
  - Later, cloud "combines" data encrypted under different keys: $Enc_{pk1,\ldots,pkt}(f(m_1,\ldots,m_t)) \leftarrow Eval(pk_1,\ldots pk_t,f,c_1,\ldots c_t)$.
- FHE doesn't do this "automatically".
- But, [LATV12]: "On-the-fly Multiparty Computation on the Cloud via Multikey FHE":
  - They have a scheme that does this.

That's It for Homomorphic Encryption Basics…

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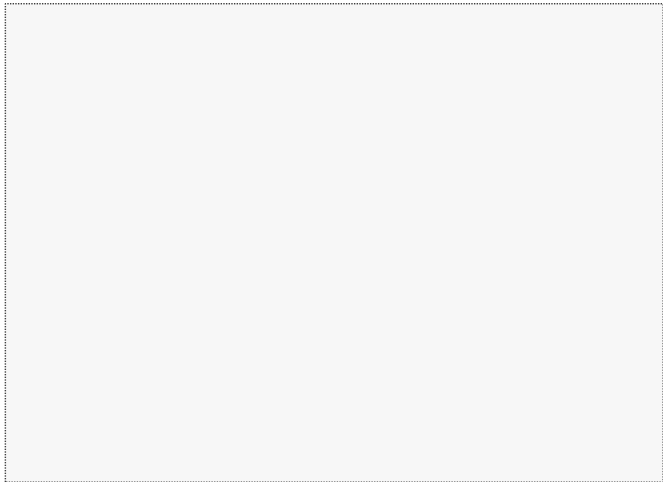
- Now, all we need is an encryption scheme that:
  - Given any encryptions $E(b_1)$ and $E(b_2)$,
  - can output encryptions $E(b_1+b_2)$ and $E(b_1 \times b_2)$,
  - forever,
  - without using the secret key of course.
- Pre-2009 schemes were *somewhat homomorphic*.
  - They could do ADD *or* MULT, not both, indefinitely.
  - Analogous to a glovebox with "clumsy" gloves.



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Somewhat Homomorphic Encryption (SWHE)

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SWHE: What's it Good For?

- I thought we were doing FHE…

Why Somewhat HE?

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- Performance!
  - For many somewhat simple functions, the “overhead” of SWHE is much less than overhead of FHE
  - “Overhead” = (time of encrypted computation)/(time of unencrypted computation)
- Stepping-stone to FHE
  - Most FHE schemes are built “on top of” a SWHE scheme with special properties.

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SWHE: Performance

FHE Implementations

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- First attempt [Smart-Vercauteren 2010]
  - Implemented (a variant of) the underlying SWHE
  - But parameters too small to get bootstrapping
- Second attempt [Gentry-Halevi 2011a]
  - Implemented a similar variant
  - Many more optimizations, tradeoffs
  - Could implement the complete FHE for 1st time

Gentry-Halevi Implementation [GH11a]

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- Using NTL/GMP
- Run on a "strong" 1-CPU machine
  - Xeon E5440 / 2.83 GHz (64-bit, quad-core) 24 GB memory
- Generated/tested instances in 4 dimensions:
  - Toy(2^9), Small(2^{11}), Med(2^{13}), Large(2^{15})
- Details at https://researcher.ibm.com/researcher/view_project.php?id=1548

Performance: SWHE

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Dimension	KeyGen	Enc amortized	Mult / Dec	degree
2048 800,000-bit integers	1.25 sec	.060 sec	.023 sec	~200
8192 3,200,000-bit integers	10 sec	.7 sec	.12 sec	~200
32768 13,000,000-bit integers	95 sec	5.3 sec	.6 sec	~200

PK is 2 integers, SK one integer

Performance: FHE

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Dimension	KeyGen	PK size	ReCrypt
2048	40 sec	70 MByte	31 sec
8192	8 min	285 MByte	3 min
32768	2 hours	2.3 GByte	30 minute

Can HE Be Practical? [LNV11]

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- Implementation of [BV11a] SWHE scheme.
- For lattice dim. 2048, Mult takes 43 msec.
  - Comparable to 23 msec of [GH10]
  - They use Intel Core 2 Duo Processor at 2.1 GHz.
- Shows lattice-based SWHE can compute quadratic functions more efficiently than [BGN05].

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SWHE: Applications

SWHE Evaluates Low Degree

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- Rule of Thumb: If your function f can be expressed as a low-degree polynomial, SWHE might be sufficient.

SWHE Evaluates Low Degree

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- Private information retrieval
  - Client wants bit B_i of database $B_1 \ldots B_n$, w/o revealing i.
  - The PIR function has degree only log n.
  - Easily achievable with SWHE.

SWHE Evaluates Low Degree

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- Keyword Search / String Matching
  - Client wants to know whether encrypted string $s = s_1 \ldots s_m$ is in one of its encrypted files
  - Comparison of two m-bit strings is a m-degree poly.
  - OR of n comparisons is a n-degree poly.
  - "Smolensky trick": in both cases we can reduce the degree to k, with a 2^{-k} probability of error.

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SWHE: Stepping-Stone to FHE

- Tomorrow, we'll see how SWHE helps construct FHE…

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SWHE: Older Schemes

- RSA, ElGamal, Paillier, Boneh-Goh-Nissim, Ishai-Paskin, …

  I won't cover these.

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SWHE: To The Constructions!!!

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Polly Cracker: An Early Attempt at SWHE

- And perhaps the most "natural" way to do it…

ADD and MULT, Naturally…

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Most Natural Approach
Ciphertexts live in a "ring".
ADDing ciphertexts (as ring elements) adds underlying plaintexts.
Some for MULT.

- Definition of (commutative) ring:
  - Like a field, without inverses.
  - It has $+$, $\times$, 0 and 1, additive and multiplicative closure.
- Examples: integers Z, polynomials Z[x,y,…], …

Polly Cracker

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Main Idea

Encryptions of 0 are polynomials that evaluate to 0 at the secret key.

- KeyGen: Secret = some point $(s_1, \ldots, s_n) \in Z_q^n$.
  Public key: Polys $\{f_i(x_1,\ldots,x_n)\}$ s.t. $f_i(s_1,\ldots,s_n)=0 \bmod q$.
- Encrypt: From $\{f_i\}$, generate *random* polynomial g s.t. $g(s_1,\ldots,s_n) = 0 \bmod q$. Ciphertext is: $c(x_1,\ldots,x_n) = m + g(x_1,\ldots,x_n) \bmod q$.
- Decrypt: Evaluate ciphertext at the secret: $c(s_1,\ldots,s_n) = m \bmod q$.
- ADD and MULT: Output sum or product of ciphertext polynomials.

Polly Cracker

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Main Idea

Encryptions of 0 are polynomials that evaluate to 0 at the secret key.

- Semantic Security (under chosen plaintext attack): Given two ciphertexts c_0 and c_1, can you distinguish whether:
  - c_0 and c_1 encrypt same message?
  - $c_0 - c_1$ encrypts 0?
  - $c_0 - c_1$ evaluates to 0 at secret key?
  - Solve "Ideal Membership" Problem?

Ideals: Definition and Examples

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- Ideal: Subset I of a ring R that is:
  - Additively closed: $i_1, i_2 \in I \rightarrow i_1+i_2 \in I$.
  - Closed under mult with R: $i \in I, r \in R \rightarrow i \cdot r \in I$.
- Example:
  - R = Z, the integers. I = (5), multiples of 5.
  - R = Z[x,y]. I = {f(x,y) ∈ Z[x,y]: f(7,11) = 0}.
    - I = (x−7,y−11). These "generate" the ideal.
- "Modulo"
  - 7 modulo (5) = 2, or 7 ∈ 2+(5)
  - g(x,y) modulo (x−7,y−11) = g(7,11).

Back to Polly Cracker…

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Main Idea

Encryptions of 0 are polynomials that evaluate to 0 at the secret key.

- Semantic Security: Ideal Membership Problem:
  - Given ciphertext polys $c_1(x_1,\ldots,x_n)$ and $c_2(x_1,\ldots,x_n)$,
  - Distinguish whether $c_1(x_1,\ldots,x_n)-c_2(x_1,\ldots,x_n)$ is in the ideal $(x_1-s_1, \ldots, x_n-s_n)$.

Polly Cracker Cryptanalysis

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- [AFFP11] Sadly, Polly Cracker is typically easy to break, using just linear algebra.
- Public key: polys $\{f_i\}$ such that $f_i(s_1,\ldots,s_n)=0$.
- Computing Grobner bases is hard, *in general.*
- *In practice*, only a small (polynomial #) of monomials can be used in the ciphertexts.

Polly Cracker Cryptanalysis

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- An Attack:
  - Collect lots of encryptions $\{c_i\}$ of 0.
    - (These are elements of an ideal I.)
  - The c_i's generate a lattice L (over the multivariate monomials). Compute Hermite Normal Form (HNF) of L.
  - To break semantic security, reduce $c_1 - c_2$ mod HNF(L): the result will be 0 if $m_1 = m_2$.

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Noisy Polly Cracker

- Adding noise to Polly Cracker to defeat attacks…

Polly Cracker

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Main Idea

Encryptions of 0 are polynomials that evaluate to 0 at the secret key.

Noisy Polly Cracker [AFFP11]

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Main Idea

Encryptions of 0 are polynomials that evaluate to something small and even (smeven) 0 at the secret key.

- KeyGen: Secret = some point $(s_1, \ldots, s_n) \in Z_q^n$.
  Public key: $\{f_i(x_1,\ldots,x_n)\}$ s.t. $f_i(s_1,\ldots,s_n)=2e_i \bmod q$, $|e_i| \ll q$.
- Encrypt: Generate random poly g s.t. $g(s_1,\ldots,s_n)=$ smeven from $\{f_i\}$. Ciphertext is $c(x_1,\ldots,x_n) = m + g(x_1,\ldots,x_n) \bmod q$ for message $m \in \{0,1\}$.
- Decrypt: $c(s_1,\ldots,s_n) = m+$smeven $\bmod q$. Reduce mod 2.
- ADD and MULT: Output sum or product of ciphertext polys.

Noisy Polly Cracker [AFFP11]

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Main Idea

Encryptions of 0 are polynomials that evaluate to something small and even (smeven) 0 *modulo* a secret ideal.

- KeyGen: Secret ideal = $(x_1-s_1, \ldots, x_n-s_n)$.
  Public key: $\{f_i(x_1,\ldots,x_n)\}$ s.t. $f_i(s_1,\ldots,s_n)=2e_i \bmod q$, $|e_i| \ll q$.
- Encrypt: Generate random poly g s.t. $g(s_1,\ldots,s_n)=$ smeven from $\{f_i\}$. Ciphertext is $c(x_1,\ldots,x_n) = m + g(x_1,\ldots,x_n) \bmod q$ for message $m \in \{0,1\}$.
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Noisy Polly Cracker [AFFP11]

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Main Idea

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- KeyGen: Secret ideal = $(x_1-s_1, \ldots, x_n-s_n)$.
  Public key: $\{f_i(x_1,\ldots,x_n)\}$ s.t. $f_i(s_1,\ldots,s_n)=2e_i \bmod q$, $|e_i| \ll q$.
- $g(s_1,\ldots,s_n)=\text{smeven}$
  $m + g(x_1,\ldots,x_n) \bmod q$

We call $c(s_1,\ldots,s_n)]_q$ the "noise" of the ciphertext.

ADDs and MULTs make the "noise" grow.

- Decrypt: $c(s_1,\ldots,s_n) = m+\text{smeven} \bmod q$. Reduce mod 2.
- ADD and MULT: Output sum or product of ciphertext polys.

Noisy Ciphertexts

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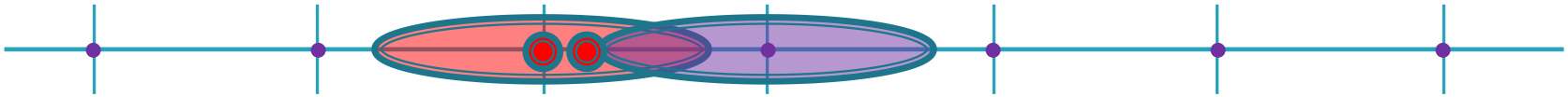

- Each ciphertext has some noise that hides the message.
- Think: “hidden” error correcting codes…
- If error is small, Alice can use knowledge of “hidden” code, or a (hidden) good basis of a known code to remove the noise.
- If noise is large, decryption becomes hopeless even for Alice.

Adding and Multiplying Noise

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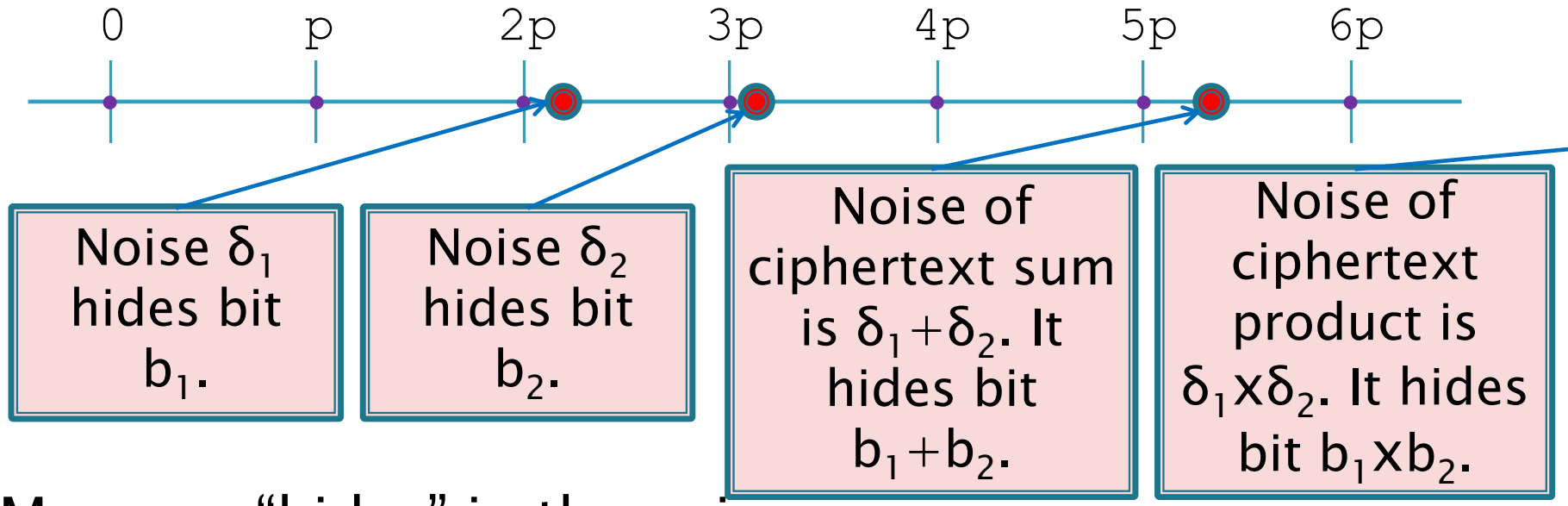

- Message "hides" in the noise.
- Adding ciphertexts adds the noises.
- Multiplying ciphertexts multiplies the noises.
- The ciphertext noisiness grows!
  - Eventually causes a decryption error!

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SWHE over the Integers [vDGHV10]

- Maybe the simplest SWHE scheme you could imagine…

A Symmetric SWHE Scheme [vDGHV10]

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- Shared secret key: odd number p
- To encrypt a bit m in $\{0,1\}$:
  - Choose at random small $r \ll p$, large q
  - Output $c = m + 2r + pq$
    - Ciphertext is close to a multiple of p
    - m = LSB of distance to nearest multiple of p
- To decrypt c:
  - Output $m = (c \bmod p) \bmod 2 = [[c]_p]_2$
- ADD, MULT: Output $c \leftarrow c_1 + c_2$ or $c \leftarrow c_1 \times c_2$.

What could be Simpler?

A Symmetric SWHE Scheme [vDGHV10]

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- Shared secret key: odd number p
- To encrypt a bit m in $\{0,1\}$:
  - Choose at random small $r \ll p$, large q
  - Output $c = m + 2r + pq$
    - Ciphertext is close to a multiple of p
    - m = LSB of distance to nearest multiple of p
- To decrypt c:
  - Output $m = (c \bmod p) \bmod 2 = [[c]_p]_2$
- ADD, MULT: Output $c \leftarrow c_1 + c_2$ or $c \leftarrow c_1 \times c_2$.

(p) is our secret ideal.

An encryption of 0 is small and even modulo our ideal.

To decrypt, evaluate c modulo the ideal. Then reduce mod 2.

Asymmetric SWHE [vDGHV10]

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- Secret key is an odd p as before
- Public key pk has "encryptions of 0" $x_i = 2r_i + q_i p$
  - Actually $x_i = [2r_i + q_i p]_{x_0}$ for $i = 1, \ldots, n$.
- Enc(pk, m) = m+subset-sum(x_i's)
  - Actually, Enc(pk, m) = $[m + \text{subset-sum}(x_i\text{'s}) + 2r]_{x_0}$.
- Dec(sk, c) = $[[c]_p]_2$

Making a public key out of "encryptions of 0" formalized by Rothblum ("From Private Key to Public Key", TCC'11).

Asymmetric SWHE [vDGHV10]

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- Secret key is an odd p as before
- Public key pk has "encryptions of 0" $x_i = 2r_i + q_i p$
  - Actually $x_i = [2r_i + q_i p]_{x_0}$ for $i = 1, \ldots, n$.
- Enc(pk, m) = m+subset-sum(x_i's)
  - Actually, Enc(pk, m) = $[m + \text{subset-sum}(x_i\text{'s}) + 2r]_{x_0}$.
- Dec(sk, c) = $[[c]_p]_2$

Quite similar to Regev's '03 scheme. Main difference: SWHE uses much more aggressive parameters…

Security

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- Approximate GCD (approx-gcd) Problem:
  - Given many $x_i = s_i + q_i p$, output p
  - Example params: $s_i \sim 2^{O(\lambda)}$, $p \sim 2^{O(\lambda^2)}$, $q_i \sim 2^{O(\lambda^5)}$, where λ is security parameter
    - Best known attacks (lattices) require 2^{λ} time
- Reduction:
  - If approx-gcd is hard, scheme is semantically secure

Hardness of Approximate-GCD

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- Several lattice-based approaches for solving approximate-GCD
  - Studied in [Howgrave-Graham01], more recently in [vDGV10, CH11, CN11]
  - All run out of steam when $|q_i| \gg |p|^2$, where $|p|$ is number of bits of p
  - In our case $|p| = O(\lambda^2)$, $|q_i| = O(\lambda^5) \gg |p|^2$

Relation to Simultaneous Diophantine Approximation

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- $x_i = q_i p + r_i$ ($r_i \ll p \ll q_i$), $i = 0,1,2,\ldots$
  - $y_i = x_i/x_0 = (q_i+s_i)/q_0$, $s_i \sim r_i/p \ll 1$
  - $y_1, y_2, \ldots$ is an instance of SDA
    - q_0 is a good denominator for all y_i's
- Use Lagarias's algorithm:
  - Consider the rows of this matrix:
  - Find a short vector in the lattice that they span
  - $\langle q_0,q_1,\ldots,q_t\rangle \cdot L$ is short
  - Hopefully we will find it.

$$L = \begin{pmatrix} R & x_1 & x_2 & \ldots & x_t \\ -x_0 & & & & \\ & -x_0 & & & \\ & & & \ldots & \\ & & & & -x_0 \end{pmatrix}$$

Relation to SDA (cont.)

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- When will Lagarias' algorithm succeed?
  - $<q_0,q_1,\ldots,q_t>\cdot L$ should be shortest in lattice
    - In particular shorter than $\sim\det(L)^{1/t+1}$
  - This only holds for $t > \log Q/\log P$
  - The dimension of the lattice is $t+1$
  - Rule of thumb: takes $2^{t/k}$ time to get 2^k approximation of SVP/CVP in lattice of dim t.
    - $2^{|q_0|/|p|^2} = 2^{\lambda}$ time to get $2^{|p|} \gg 2^{\lambda}$ approx.

Minkowski bound

- Bottom line: no known efficient attack on approx-gcd

How Homomorphic Is It?

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- Suppose $c_1 = m_1 + 2r_1 + q_1 p$, …, $c_t = m_t + 2r_t + q_t p$
- ADD: $c = c_1 + c_2$.
  - Noise of c is $[c]_p = (m_1 + m_2 + 2r_1 + 2r_2)$, sum of noises
- MULT: $c = c_1 \times c_2$.
  - Noise of c is $[c]_p = (m_1 + 2r_1) \times (m_2 + 2r_2)$, product of noises.
- f: $c = f(c_1, \ldots, c_t) = f(m_1 + 2r_1, \ldots, m_t + 2r_t)$, the function f applied to the noises.

How Homomorphic Is It?

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- Claim: If $|f(m_1+2r_1, \ldots, m_t+2r_t)| < p/2$ for all possible "fresh" noises m_i+2r_i, the SWHE scheme can Eval f correctly.
- Proof:
  - Set $c = f(c_1, \ldots, c_t)$.
  - Then, $[c]_p = f(m_1+2r_1, \ldots, m_t+2r_t)$ by assumption.
  - Then, $[[c]_p]_2 = f(m_1, \ldots, m_t) \bmod 2$.

That's what we want!

How Homomorphic Is It?

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- What if $|f(m_1+2r_1, \ldots, m_t+2r_t)| > p/2$?
  - $c = f(c_1, \ldots, c_t) = f(m_1+2r_1, \ldots, m_t+2r_t) + qp$
    - Nearest p-multiple to c is $q'p$ for $q' \neq q$
  - $(c \bmod p) = f(m_1+2r_1, \ldots, m_t+2r_t) + (q-q')p$
  - $(c \bmod p) \bmod 2$
  - $= f(m_1, \ldots, m_t) + (q-q') \bmod 2$
  - $=$???
- We say the scheme can <u>handle</u> f if:
  - $|f(x_1, \ldots, x_t)| < p/4$
  - Whenever all $|x_i| < B$, where B is a bound on the noise of a fresh ciphertext output by Enc.

Example of a Function It Can Handle

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- Elementary symmetric poly of degree d:
  - $f(x_1, \ldots, x_t) = x_1 \cdot x_2 \cdot x_d + \ldots + x_{t-d+1} \cdot x_{t-d+2} \cdot x_t$
  - Has (t choose d) $< t^d$ monomials: a lot!!
- If $|x_i| < B$, then $|f(x_1, \ldots, x_t)| < t^d \cdot B^d$
- E can handle f if:
  - $t^d \cdot B^d < p/4 \rightarrow$ basically if: $d < (\log p)/(\log tB)$
- Example params: $B \sim 2^\lambda$, $p \sim 2^{\lambda \wedge 2}$
  - Eval can handle elem symm poly of degree about λ.

An Optimization

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- If f has degree d, $c = f(c_1, \ldots, c_t)$ will have about d times as many bits as the fresh c_i's.
- Can we reduce the ciphertext length after multiplications?

An Optimization

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- A heuristic:
  - Suppose n is bit-length of normal ciphertext.
  - Put additional "encryptions of 0" $\{y_i = 2r_i + q_i p\}$ in pk.
    - Set y_i's to increase geometrically up to square of normal ciphertext: $y_i \approx 2^{n+i}$, for i up to $\approx n$.
  - Set $c = c_1 \times c_2$ – subsetsum(y_i's), and c will have normal size.
    - Subtract off y_i's according to c's binary representation.

Performance

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- Well, a little slow…
  - Example parameters: a ciphertext is $O(\lambda^5)$ bits.
  - Least efficient SWHE scheme, asymptotically.
- But Coron, Mandal, Naccache, Tibouchi have made impressive efficiency improvements.
  - [CMNT Crypto '11]: FHE over the Integers with Shorter Public Keys
  - [CNT Eurocrypt '12]: Public-key Compression and Modulus Switching for FHE over the Integers.
  - Asymptotics are *much* better now.

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SWHE Based on LWE [BV11b]

The LWE Problem

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- Traditional Version:
  - Let χ be an error distribution.
  - Distinguish these distributions:
    - Generate uniform $s \leftarrow Z_q^n$. For many i, generate uniform $a_i \leftarrow Z_q^n$, $e_i \leftarrow \chi$, and output $(a_i, [<a_i, s>+e_i]_q)$.
    - For many i, generate uniform $a_i \leftarrow Z_q^n$, $b_i \leftarrow Z_q$ and output (a_i, b_i).

The LWE Problem

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- Noisy Polly Cracker Version:
  - Let χ be an error distribution.
  - Distinguish these distributions:
    - Generate uniform $s \leftarrow Z_q^n$. For many i, generate $e_i \leftarrow \chi$ and a linear polynomial $f_i(x_1, \ldots, x_n) = f_0 + f_1 x_1 + \ldots + f_n x_n$ (from Z_q^{n+1}) such that $[f_i(s_1, \ldots, s_n)]_q = e_i$.
    - For many i, generate and output a uniformly random linear polynomial $f_i(x_1, \ldots, x_n)$ (from Z_q^{n+1}).

Regev LWE Encryption Revisited

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- Parameters: q such that gcd(q,2)=1.
- KeyGen: Secret = uniform $\mathbf{s} \in Z_q^n$. Public key: linear polys $\{f_i(x_1,\ldots,x_n)\}$ s.t. $[f_i(\mathbf{s})]_q = 2e_i$, $|e_i| \ll q$.
- Encrypt: Set $g(x_1,\ldots,x_n)$ as a random subset sum of $\{f_i(x_1,\ldots,x_n)\}$. Output $c(x_1,\ldots,x_n) = m + g(x_1,\ldots,x_n)$.
- Decrypt: $[c(\mathbf{s})]_q = m + \text{smeven}$. Reduce mod 2.
- Security:
  - Public key consists of an LWE instance, doubled.
  - Leftover hash lemma.

SWHE Based on LWE [BV11b]

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- Parameters: q such that gcd(q,2)=1.
- KeyGen: Secret = uniform $\mathbf{s} \in Z_q^n$. Public key: linear polys $\{f_i(x_1,\ldots,x_n)\}$ s.t. $[f_i(\mathbf{s})]_q = 2e_i$, $|e_i| \ll q$.
- Encrypt: Set $g(x_1,\ldots,x_n)$ as a random subset sum of $\{f_i(x_1,\ldots,x_n)\}$. Output $c(x_1,\ldots,x_n)=m+g(x_1,\ldots,x_n)$.
- Decrypt: $[c(\mathbf{s})]_q = m+\text{smeven}$. Reduce mod 2.
- ADD and MULT:
  - Output sum or product of ciphertext polynomials.

Relinearization [BV11b]

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- After MULT, we have ciphertext $c(\mathbf{x}) = c_1(\mathbf{x}) \cdot c_2(\mathbf{x})$ that encrypts some m under key **s**.
  - $[c(\mathbf{s})]_q = m + \text{smeven}$
  - $c(\mathbf{x})$ is a *quadratic* poly with $O(n^2)$ coefficients.
- What we want: a *linear* ciphertext $d(\mathbf{y})$ that encrypts same m under some key $\mathbf{t} \in Z_q^n$.
- Relinearization maps a long quadratic ciphertext under **s** to a normal linear ciphertext under **t**.

Relinearization: From Quadratic to Linear (A Change of Variables)

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- First step: View $c(\mathbf{x})$ as a *long* linear ciphertext $C(\mathbf{X})$.
  - Set the variables $X_{ij} = x_i \cdot x_j$.
  - Set the values $S_{ij} = s_i \cdot s_j$.
  - Set $C(\mathbf{X}) = \sum c_{1i} c_{2j} X_{ij}$.
  - Then, $[C(\mathbf{S})]_q = [c(\mathbf{s})]_q = m + \text{smeven}$.
  - (This is only a change of perspective.)

Second Step: Key Switching

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- Input: *Long* linear ciphertext C(**X**) with N > n, where $[C(\mathbf{S})]_q = e = m+smeven$, and $\mathbf{S} = (S_1,\ldots, S_N)$ is a long secret key.
- Output: *Normal-length* linear ciphertext d(**x**), where $[d(\mathbf{t})]_q = e+smeven = m+smeven$, and $\mathbf{t} = (t_1,\ldots, t_n)$ is a normal-length secret key.
- Special case: $N \approx n^2$.

Key Switching Details

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- SwitchKeyGen(**S**,**t**): Output linear polys $\{h_i(\mathbf{x})\}$, $i \in \{1,\ldots,N\}$ such that:
  $[h_i(\mathbf{t})]_q = S_i + \text{smeven}_i$
  (like an encryption of S_i under **t**)
  Add Aux(**S**,**t**) = $\{h_i(\mathbf{x})\}$ to pk.
- SwitchKey(pk, C(**X**)): Set $d(\mathbf{x}) = \sum_i C_i \cdot h_i(\mathbf{x})$.
- $d(\mathbf{t}) = \sum_i C_i \cdot (S_i + \text{smeven}_i) = C(\mathbf{S}) + \sum_i C_i \cdot \text{smeven}_i$
- Oh wait, $\sum_i C_i \cdot \text{smeven}_i$ is not small and even…
- Fix: Bit-decompose C first so that it has small coefficients…

Key Switching: Bit Decomposition Interlude

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- BitDecomp:
  - Let BitDecomp(C(**X**)) be the bit-decomposition of C(**X**).
  - $(U_1(\mathbf{X}),\ldots, U_{\log q}(\mathbf{X})) \leftarrow$ BitDecomp(C(**X**)), where each $U_j(\mathbf{X})$ has 0/1 coefficients and $C(\mathbf{X}) = \sum_j 2^j \cdot U_j(\mathbf{X})$.
- Powerof2:
  - $(\mathbf{S}, 2\mathbf{S}, \ldots, 2^{\log q}\mathbf{S}) \leftarrow$ Powersof2(**S**).
- Let **C'**=BitDecomp(**C**) and **S'** = Powerof2(**S**). Then, $\langle \mathbf{C'},\mathbf{S'}\rangle = \langle \mathbf{C},\mathbf{S}\rangle$.
- So, **C'**(**S'**) = **C**(**S**) mod q.

Key Switching Details

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- SwitchKeyGen(**S**,**t**): Output linear polys $\{h_i(\mathbf{x})\}$, $i \in \{1,\ldots,N\}$ such that:
  $[h_i(\mathbf{t})]_q = S_i' + \text{smeven}_i$
  (like an encryption of S_i' under **t**)
  Add Aux(**S'**,**t**) = $\{h_i(\mathbf{x})\}$ to pk.
- SwitchKey(pk, $C'(\mathbf{X})$): Set $d(\mathbf{x}) = \sum_i C_i' \cdot h_i(\mathbf{x})$.
- $d(\mathbf{t}) = \sum_i C_i' \cdot (S_i' + \text{smeven}_i) = C'(\mathbf{S'}) + \sum_i C_i' \cdot \text{smeven}_i$

- Now, $\sum_i C_i' \cdot \text{smeven}_i$ is small and even…

Key Switching: Summary

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- Functionality:
  - Regev ciphertext under key **S** → Ciphertext under **t**.
  - Need to put Aux(**S**,**t**) in pk.
  - Like proxy re-encryption.
  - Relinearization is only a special case.
    - Later, we will use key switching in a different context.
- Effect on noise: SwitchKey increases noise only additively.
- For depth L circuit, use a chain of L encrypted secret keys.

SWHE from LWE [BV11b]: Summary

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- Follows Noisy Polly Cracker blueprint
  - With a relinearization step.
- Relinearization / key-switching
  - Doesn't increase the noise much.
  - So noise analysis, and "homomorphic capacity" analysis, is similar to integer scheme.
  - For L depth circuit, use a chain of L encrypted secret keys.

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SWHE Based on Ideal Lattices [Gen09]

- I'll skip my 2009 scheme, and focus on RLWE- and NTRU-based schemes.

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SWHE Based on RLWE [BV11a]

The Ring-LWE Problem

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- Traditional Version:
  - Let χ be an error distribution over $R = Z_q[y]/(y^n+1)$.
  - Distinguish these distributions:
    - Generate uniform $s \leftarrow R$. For many i, generate uniform $a_i \leftarrow R$, $e_i \leftarrow \chi$, and output $(a_i, a_i \cdot s + e_i)$.
    - For many i, generate uniform $a_i \leftarrow R$, $b_i \leftarrow R$ and output (a_i, b_i).

The Ring-LWE Problem

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- Noisy Polly Cracker Version:
  - Let χ be an error distribution over $R = Z_q[y]/(y^n+1)$.
  - Distinguish these distributions:
    - Generate uniform $s \leftarrow R$. For many i, generate $e_i \leftarrow \chi$ and a linear polynomial $f_i(x) = f_0+f_1x$ (from R^2) such that $f_i(s) = e_i$.
    - For many i, generate and output a uniformly random linear polynomial $f_i(x)$ (from R^2).

[LPR10] RLWE-Based Encryption

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- Parameters: q with gcd(q,2)=1, $R = Z_q[y]/(y^n+1)$.
- KeyGen: Secret = uniform $s \in R$. Public key: linear polys $\{f_i(x)\}$ s.t. $f_i(s)=2e_i$, $|e_i| \ll q$.
- Encrypt: Set g(x) as a random subset sum of $\{f_i(x)\}$. Output c(x)=m+g(x).
  - m can be a "polynomial", an element of $Z_2[y]/(y^n+1)$.
- Decrypt: c(s) = m+smeven. Reduce mod 2.

SWHE from RLWE [BV11a]

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- Parameters: q with gcd(q,2)=1, $R = Z_q[y]/(y^n+1)$.
- KeyGen: Secret = uniform $s \in R$. Public key: linear polys $\{f_i(x)\}$ s.t. $f_i(s)=2e_i$, $|e_i| \ll q$.
- Encrypt: Set g(x) as a random subset sum of $\{f_i(x)\}$. Output c(x)=m+g(x).
  - m can be a "polynomial", an element of $Z_2[y]/(y^n+1)$.
- Decrypt: c(s) = m+smeven. Reduce mod 2.

- ADD and MULT: Add or multiply the ciphertext polynomials.

Relinearization [BV11b] applied to [BV11a]

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- After MULT, we have ciphertext $c(x) = c_1(x) \cdot c_2(x)$ that encrypts some m under key s.
  - $c(s) = m + s^{meven}$
  - c(x) is a *quadratic* poly with 3 coefficients.
- What we want: a *linear* ciphertext d(x) that encrypts same m under some key $t \in R$.
- Relinearization maps a long quadratic ciphertext under s to a normal linear ciphertext under t.

Relinearization: From Quadratic to Linear (A Change of Variables)

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- First step: View c(x) as a *long* linear ciphertext C(**X**).
  - Set the variables $X_1 = x$ and $X_2 = x^2$.
  - Set the values $S_1 = s$ and $S_2 = s^2$.
  - Set $C(\mathbf{X})=(c_{11}x+c_{10})(c_{21}x+c_{20})= c_{11}c_{21}X_2+(c_{11}c_{20}+c_{10}c_{21})X+c_{10}c_{20}$.
  - Then, $C(\mathbf{S}) = c(s) = m+s^m_{even}$.
  - (This is only a change of perspective.)

Second Step: Key Switching

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- Input: *Long* linear ciphertext C(**X**), where C(**S**) = e = m+smeven, and **S** = (S_1,S_2) is a long secret key.
- Output: *Normal-length* linear ciphertext d(x), where d(t) = e+smeven = m+smeven, and t $\in$ R.

Key Switching Details

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- SwitchKeyGen(**S**,t): Output linear polys $\{h_i(x)\}$, $i \in \{1,\ldots,N\}$ such that:
  $h_i(t) = S_i + \text{smeven}_i$
  (like an encryption of S_i under t)
  Add Aux(**S**,t) $= \{h_i(x)\}$ to pk.
- SwitchKey(pk, C(**X**)): Set $d(x) = \sum_i C_i \cdot h_i(x)$.
- $d(t) = \sum_i C_i \cdot (S_i + \text{smeven}_i) = C(\mathbf{S}) + \sum_i C_i \cdot \text{smeven}_i$
- Oh wait, $\sum_i C_i \cdot \text{smeven}_i$ is not small and even…
- Fix: Bit-decompose C first so that it has small coefficients…

Key Switching: Bit Decomposition Interlude

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- BitDecomp:
  - Let BitDecomp(C(**X**)) be the bit-decomposition of C(**X**).
  - $(U_1(\mathbf{X}),\ldots, U_{\log q}(\mathbf{X})) \leftarrow$ BitDecomp(C(**X**)), where each $U_j(\mathbf{X})$ has coefficients (in R) that are 0/1 polynomials and $C(\mathbf{X}) = \sum_j 2^j \cdot U_j(\mathbf{X})$.
- Powerof2:
  - $(\mathbf{S}, 2\mathbf{S}, \ldots, 2^{\log q}\,\mathbf{S}) \leftarrow$ Powersof2(**S**).
- Let **C'**=BitDecomp(**C**) and **S'** = Powerof2(**S**). Then, $\langle \mathbf{C'},\mathbf{S'}\rangle = \langle \mathbf{C},\mathbf{S}\rangle$.
- So, **C'**(**S'**) = **C**(**S**) in R.

Key Switching Details

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- SwitchKeyGen(**S**,t): Output linear polys $\{h_i(\mathbf{x})\}$, $i \in \{1,\ldots,N\}$ such that:
  $h_i(t) = S_i' + \text{smeven}_i$
  (like an encryption of S_i' under t)
  Add Aux(**S'**,t) $= \{h_i(x)\}$ to pk.
- SwitchKey(pk, $C'(\mathbf{X})$): Set $d(x) = \sum_i C_i' \cdot h_i(x)$.
- $d(t) = \sum_i C_i' \cdot (S_i' + \text{smeven}_i) = C'(\mathbf{S'}) + \sum_i C_i' \cdot \text{smeven}_i$

- Now, $\sum_i C_i' \cdot \text{smeven}_i$ is small and even…

RLWE Key Switching: Summary

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- Functionality: as in LWE.
- Effect on noise: SwitchKey increases noise only additively, as in LWE.
- Performance: Better!
  - RLWE:
    - Key switching involves O(log q) multiplications in R.
    - We can use FFT for multiplication.
    - quasi-O(n log q) work
  - LWE:
    - Relinearization is $O(n^3 \log q)$ work.

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NTRU-Based SWHE

NTRU-Based SWHE ([LATV12] and [GHLPSS12])

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- Parameters: q with gcd(q,2)=1, $R = Z_q[y]/(y^n+1)$.
- KeyGen: Secret = uniform $s \in R$. Public key: linear polys $\{f_i(x)\}$ s.t. $f_i(s)=2e_i$, $|e_i| \ll q$. More reqs:
  - s is small and 1 mod 2 (smodd?)
  - $f_i(x)$ has no constant term – i.e., $f_{i1} \cdot s = 2e_i$.
- Encrypt: Set g(x) as a random subset sum of $\{f_i(x)\}$. Output $c(x)=m \cdot x+g(x)$.
  - m can be a "polynomial", an element of $Z_2[y]/(y^n+1)$.
- Decrypt: $c(s) = m \cdot s+$smeven. Reduce mod 2.

- Security: NTRU Problem: Do f_{i1}'s have form $f_{i1}=2e_i/s_i$; e_i, s_i short?

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- Decrypt: $c(s) = m \cdot s+$smeven. Reduce mod 2.

- ADD and MULT: Add or multiply the ciphertext polynomials.

NTRU-Based SWHE: Multiplication Becomes Simpler

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- Multiplicands: $c_1(x) = c_{11} \cdot x$ and $c_2(x) = c_{21} \cdot x$.
- Product: $c(x) = c_1(x) \cdot c_2(x) = c_{11} \cdot c_{21} \cdot x^2$.
- Can we forget key switching?
  - Just view $t = s^2$ as the new secret key.
  - $c(t) = m_1 \cdot m_2 \cdot t + smeven = m_1 \cdot m_2 + smeven$.
- Not quite: What if we want to add a ciphertext under key s to another ciphertext under s^2?

NTRU-Based SWHE: Key Switching Becomes Simpler

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- Multiplicands: $c_1(x) = c_{11} \cdot x$ and $c_2(x) = c_{21} \cdot x$.
- Product: $c(x) = c_1(x) \cdot c_2(x) = c_{11} \cdot c_{21} \cdot x^2$.
- Aux(S,t): Choose $e^* \leftarrow \chi$, and set $e_{S,t} = 2e^*+1$. Output $a_{S,t} = S \cdot e_{S,t} \cdot t^{-1}$. ($e_{S,t} \cdot t^{-1}$ should look random.)
- SwitchKey(c,$a_{S,t}$):
  - Suppose $c \cdot S = e = m + 2e_{even}$.
  - New ciphertext is $c' = c \cdot a_{S,t}$.
  - Then, $c' \cdot t = (c \cdot a_{S,t})t = c(a_{S,t} \cdot t)$
    $= c(S \cdot e_{S,t}) = e \cdot e_{S,t} = m + 2e_{even}$.
- Noise increases multiplicatively.

NTRU-Based SWHE: MultiKey

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- Two ciphertexts under different keys:
  - $c_1(x) = c_{11} \cdot x$ and $c_2(x) = c_{21} \cdot x$.
  - $c_1(s_1) = m_1 \cdot s_1 + \text{smeven}$, $c_2(s_2) = m_2 \cdot s_2 + \text{smeven}$.
- Product: $c_{11}c_{21}s_1s_2 = m_1m_2s_1s_2 + \text{smeven} = m_1m_2 + \text{smeven}$.
- [LATV12]: Cloud can (noninteractively) combine data encrypted under different keys.

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Other SWHE Schemes

- Insert your scheme here!

Thank You! Questions?

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