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Fully Homomorphic Encryption

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Based Mostly on [van-Dijk, Gentry, Halevi, Vaikuntanathan, EC 2010]

[redacted]

Part I

Somewhat Homomorphic Encryption

Motivating Application: Simple Keyword Search

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- **Storing an encrypted file F on a remote server**
- **Later send keyword w to server, get answer, determine whether F contains w**
  - Trivially: server returns the entire encrypted file
  - We want: answer length independent of |F|

Claim: to do this, sufficient to evaluate low-degree polynomials on encrypted data

- degree ~ security parameter

Protocol for keywork-search

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- File is encrypted bit by bit, $E(F_1)$ $E(F_2)$ … $E(F_t)$
- Word has s bits $w_1w_2…w_s$
- For i=1,2,…,t-s+1, server computes the bit
  $c_i = \prod_{j=1}^{s}(1+w_j+F_{i+j-1}) \bmod 2$
  - $c_i=1$ if w appears at position i, else $c_i=0$
  - Each c_i is a degree-s polynomial in the F_i's
    - Trick from [Smolansky'93] to get degree-n polynomials, error-probability 2^{-n}
- Return n random subset-sums of the c_i's (mod 2) to client
  - Still degree-n, another 2^{-n} error

Computing low-degree polynomials on ciphertexts

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- Want an encryption scheme (Gen, Enc, Dec)
  - Say, symmetric bit-by-bit encryption
  - Semantically secure, $E(0) \approx E(1)$
- Another procedure: $\mathrm{Eval}(f, C_1, \ldots C_t)$
  - f is a binary polynomial in t variables, degree$\leq$n
    - Represented as arithmetic circuit
  - The C_i's are ciphertexts
- For any such f, and any $C_i = \mathrm{Enc}(x_i)$ it holds that $\mathrm{Dec}(\ \mathrm{Eval}(f, C_1, \ldots C_t)\ ) = f(x_1, \ldots, x_t)$
  - Also |Eval(f,…)| does not depend on the “size” of f (i.e., # of vars or # of monomials, circuit-size)
  - That’s called “compactness”

A Simple SHE Scheme

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- Shared secret key: odd number p
- To encrypt a bit m:
  - Choose at random small r, large q
  - Output $c = pq + \underbrace{2r + m}_{\text{The "noise"}}$
    - Ciphertext is close to a multiple of p
    - m = LSB of distance to nearest multiple of p

Noise much smaller than p

- To decrypt c:
  - Output $m = (c \bmod p) \bmod 2$

$$= c - p \cdot [[c/p]] \bmod 2$$
$$= c - [[c/p]] \bmod 2$$
$$= \mathrm{LSB}(c)\ \mathrm{XOR}\ \mathrm{LSB}([[c/p]])$$

[[c/p]] is rounding of the rational c/p to nearest integer

Why is this homomorphic?

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- **Basically because:**
  - If you add or multiply two near-multiples of p, you get another near multiple of p…

Why is this homomorphic?

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- $c_1 = q_1 p + 2r_1 + m_1$, $c_2 = q_2 p + 2r_2 + m_2$

Distance to nearest multiple of p

- $c_1 + c_2 = (q_1 + q_2)p + 2(r_1 + r_2) + (m_1 + m_2)$
  - $2(r_1 + r_2) + (m_1 + m_2)$ still much smaller than p

  ➔ $c_1 + c_2 \bmod p = 2(r_1 + r_2) + (m_1 + m_2)$

- $c_1 \times c_2 = (c_1 q_2 + q_1 c_2 - q_1 q_2)p + 2(2r_1 r_2 + r_1 m_2 + m_1 r_2) + m_1 m_2$
  - $2(2r_1 r_2 + \ldots)$ still smaller than p

  ➔ $c_1 x c_2 \bmod p = 2(2r_1 r_2 + \ldots) + m_1 m_2$

Why is this homomorphic?

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- $c_1 = q_1 p + 2r_1 + m_1, \ldots, c_t = q_t p + 2r_t + m_t$
- Let f be a multivariate poly with integer coefficients (sequence of +'s and x's)
- Let c = Eval(f, $c_1, \ldots, c_t$) = $f(c_1, \ldots, c_t)$

Suppose this noise is much smaller than p

  - $f(c_1, \ldots, c_t) = f(m_1+2r_1, \ldots, m_t+2r_t) + qp$
    $= f(m_1, \ldots, m_t) + 2r + qp$

$\rightarrow$ (c mod p) mod 2 = $f(m_1, \ldots, m_t)$

That's what we want!

How homomorphic is this?

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- **Can keep adding and multiplying until the "noise term" grows larger than p/2**
  - Noise doubles on addition, squares on multiplication
  - Multiplying d ciphertexts ➔ noise of size $\sim 2^{dn}$
- **We choose $r \sim 2^n$, $p \sim 2^{n^2}$ (and $q \sim 2^{n^5}$)**
  - Can compute polynomials of degree ~n before the noise grows too large

Keeping it small

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- **Ciphertext size grows with degree of f**
  - Also (slowly) with # of terms
- **Publish one "noiseless integer", N = pq**
  - In the symmetric setting, include N with the secret key and with every ciphertext
  - For technical reasons: q is odd, the q_i's for encryption are chosen from [q] rather than $[2^{n^5}]$
- **Ciphertext arithmetic mod N**

➔Ciphertext-size remains always the same

Public Key Encryption

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Rothblum'11: Any homomorphic and compact symmetric encryption (wrt class C including linear functions), can be turned into public key

- Still homomorphic and compact wrt essentially the same class of functions C

- Public key: N random bits $r=(r_1 \ldots r_N)$ and their symmetric encryption $c_i=Enc_{sk}(r_i)$
  - N larger than size of evaluated ciphertext
- $NewEnc_{pk}$ (b): Choose random s s.t. $\langle s,r \rangle=b$, use Eval to get $c^*=Enc_{sk}(\langle s,r \rangle)$
  - Note that s → c* is shrinking

Security

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- **The approximate-GCD problem:**
  - Input: integers $w_0, w_1, \ldots, w_t,$
    - Chosen as $w_0 = q_0 p$, $w_i = q_i p + r_i$ (p and q_0 are odd)
    - $p \in_\$ [0,P]$, $q_i \in_\$ [0,Q]$, $r_i \in_\$ [0,R]$ (with $R << P << Q$)
  - Task: find p
- **Thm: If we can distinguish Enc(0)/Enc(1) for some p, then we can find that p**
  - Roughly: the LSB of r_i is a "hard core bit"

➔ If approx-GCD is hard then scheme is secure

- **(Later: Is approx-GCD hard?)**

Hard-core-bit theorem

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A. The approximate-GCD problem:

- Input: $w_0=q_0p$, $\{w_i=q_ip+r_i\}$
  - $p\in_\$[0,P]$, $q_i\in_\$[0,Q]$, $r_i\in_\$[0,R]$ (with $R << P << Q$)
- Task: find p

B. The cryptosystem

- Input: : $N=q_0p$, $\{c_j=q_jp+r_j, LSB(r_j)\}$, $c=qp+2r+m$
  - $p\in_\$[0,P]$, $q_i\in_\$[0,Q]$, $r_i\in_\$[0,R']$ (with $R' << P << Q$)
- Task: distinguish $m=0$ from $m=1$

Thm: Solving B → solving A

- small caveat: R smaller than R'

Proof outline

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- Input: $w_0 = q_0 p$, $\{w_i = q_i p + r_i\}$
- Use the w_i's to form the c_j's and c
  - This is where we need R'>R
- Amplify the distinguishing advantage
  - From any noticeable ε to almost 1
- Use reliable distinguisher to learn q_0
  - Using the binary GCD procedure
- Finally $p = w_0/q_0$

From $\{w_i\}$ to $\{c_j, LSB(r_j)\}$

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- **We have $w_i=q_ip+r_i$, need $x_j=q_j'p+2r_j'$**
  - Then we can add the LSBs to get $c_j = x_j + m_j$
- **Set $N=w_0$, $x_j=2(\text{subsetSum}\{w_i\}+\rho_j) \bmod N$**
  - The ρ_j's are random $< R'$
- **Correctness:**
  - $\text{SubsetSum}\{r_i\}+\rho_j$ distributed almost identically to ρ_j
    - Since $R'>R$ by a super-polynomial factor
  - $2\times\text{SubsetSum}\{q_i\} \bmod q_0$ is almost random in $[q_0]$

Amplify distinguishing advantage

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- **Given *any* integer $z=qp+r$, with $r<R$:**

  Set $c = [z + m + 2(\rho + \text{subsetSum}\{w_i\})] \bmod N$
  - For random $\rho<R'$, random bit m
- **c is nearly a random ciphertext for m+LSB(r)**
  - Same reason as for the c_j's
- **$c \bmod p \bmod 2 = r+m \bmod 2$**
  - A guess for c mod p mod 2 ➔ vote for r mod 2
- **Choose many random c's, take majority**

Noticeable advantage ➔ Reliably computing r mod 2

Reliable distinguisher ➔ learning q_0

- **From *any* $z=qp+r$ ($r<R'$) can get $r \bmod 2$**
  - Note: $z = q+r \bmod 2$ (since p is odd)
  - So ($q \bmod 2$) = ($r \bmod 2$) $\oplus$ ($z \bmod 2$)
- **Given z_1, z_2, both near multiples of p**

Binary-GCD

  - Get $b_i := q_i \bmod 2$, if $z_1<z_2$ swap them
  - If $b_1=b_2=1$, set $z_1:=z_1-z_2$, $b_1:=b_1-b_2$
    - At least one of the b_i's must be zero now
  - For any $b_i=0$ set $z_i := \text{floor}(z_i/2)$
    - new-q_i = old-$q_i/2$
  - Repeat until one z_i is zero, output the other

$z = (2s)p + r$
➔ $z/2=sp+r/2$
➔ $\text{floor}(z/2) = sp+\text{floor}(r/2)$

Reliable distinguisher → learning q_0

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The odd part of the GCD

- **$z_i = q_i p + r_i$, $i=1,2$, $z' := \text{OurBinaryGCD}(z_1, z_2)$**
  - Then $z' = \text{GCD}^*(q_1, q_2) \cdot p + r'$
  - For random q,q', $\Pr[\text{GCD}(q,q')=1] \sim 0.6$
- **Try (say) $z' := \text{OurBinaryGCD}(w_0, w_1)$**
  - Hope that $z' = 1 \cdot p + r$
    - Else try again with $\text{OurBinaryGCD}(z', w_2)$, etc.
- **Then run $\text{OurBinaryGCD}(w_0, z')$**
  - The b_1 bits spell out the bits of q_0
- **Once you learn q_0, $p = w_0 / q_0$**

Is Approximate-GCD Hard?

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- **Several lattice-based approaches for solving approximate-GCD**
  - Approximate-GCD is related to Simultaneous Diophantine Approximation (SDA)
  - Studied in [Hawgrave-Graham01]
    - We considered some extensions of his attacks
- **All run out of steam when $|q_i|>|p|^2$**
  - In our case $|p|\sim n^2$, $|q_i|\sim n^5 >> |p|^2$

Relation to SDA

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- $w_0 = q_0 p$, $w_i = q_i p + r_i$ $(r_i << p << q_i)$
  - $y_i = w_i/w_0 = (q_i p + r_i)/q_0 p = (q_i + \varepsilon_i)/q_0$
    - $\varepsilon_i = r_i/p << 1$
  - $y_1, y_2, \ldots$ is an instance of SDA
    - q_0 is a denominator that approximates all y_i's
- **Try to use Lagarias'es algorithm to solve**
  - Find q_0, then $p = w_0/q_0$

Lagarias'es SDA algorithm

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- **Consider the rows of this matrix B:**
  - They span dim-(t+1) lattice
- **$(q_0,q_1,\ldots,q_t) \times B$ is short**
  - 1^{st} entry: $q_0R < Q \cdot R$
  - i^{th} entry $(i>1)$: $q_0(q_ip+r_i)-q_i(q_0p)=q_0r_i$
    - Less than $Q \cdot R$ in absolute value

  ➔ Total size less than $Q \cdot R \cdot \sqrt{t}$
    - vs. size ~$Q \cdot P$ (or more) for basis vectors
- **Hopefully we can find it with a lattice-reduction algorithm (LLL or variants)**

$$B = \begin{pmatrix} R & w_1 & w_2 & \ldots & w_t \\ & -w_0 & & & \\ & & -w_0 & & \\ & & & \ldots & \\ & & & & -w_0 \end{pmatrix}$$

Will this algorithm succeed?

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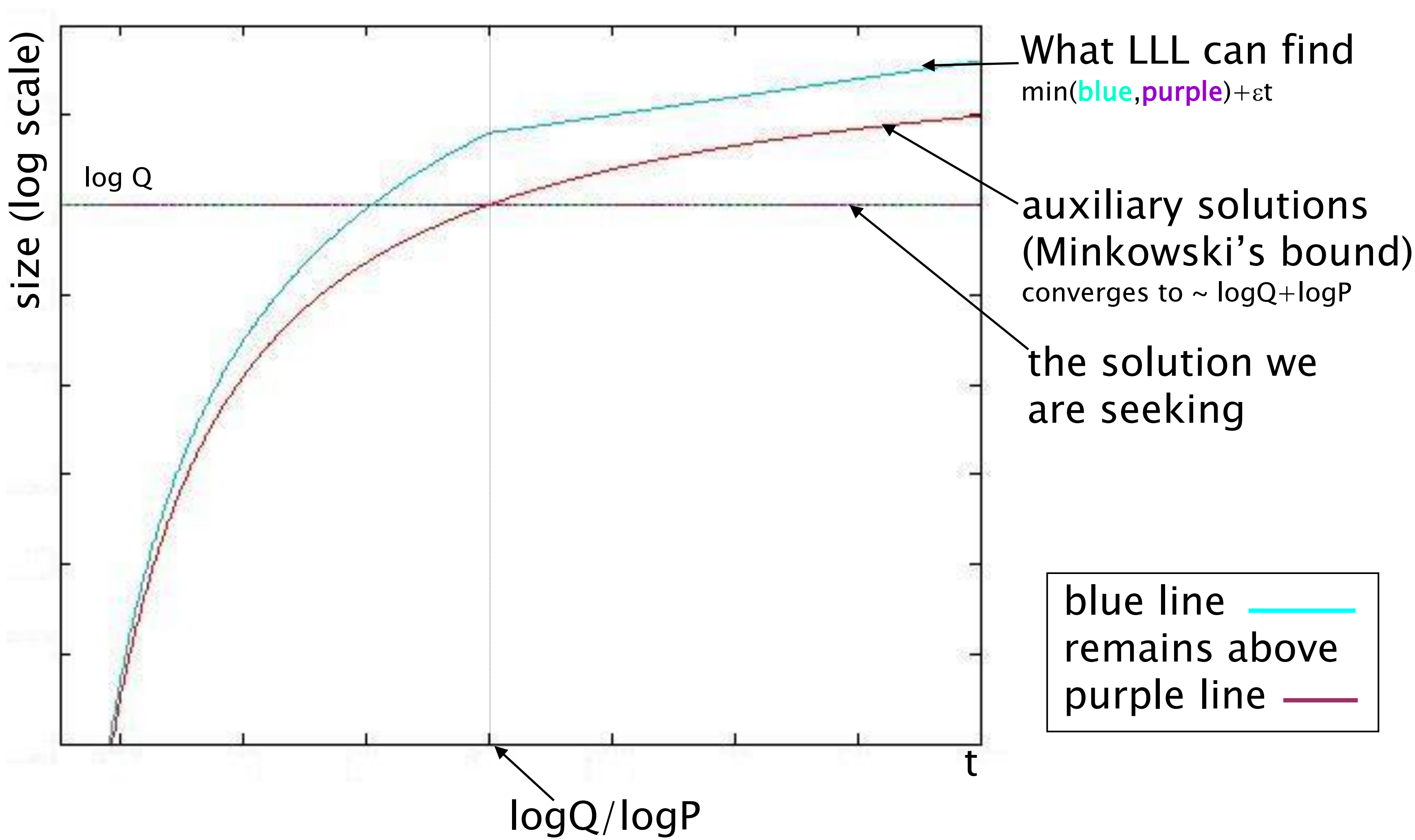
- Is $(q_0,q_1,\dots,q_t) \times B$ the shortest in the lattice?
  - Is it shorter than $\sqrt{t}\cdot\det(B)^{1/t+1}$? Minkowski bound
    - $\det(B)$ is small-ish (due to R in the corner)
  - Need $((QP)^t R)^{1/t+1} > QR$

    $\Leftrightarrow$ $t+1 > (\log Q + \log P - \log R) / (\log P - \log R)$
    $\sim \log Q/\log P$

$$\begin{pmatrix} R & w_1 & w_2 & \dots & w_t \\ & -w_0 & & & \\ & & -w_0 & & \\ & & & \dots & \\ & & & & -w_0 \end{pmatrix}$$

- $\log Q = \omega(\log^2 P)$ ➔ need $t=\omega(\log P)$
- Quality of LLL & co. degrades with t
  - Find vectors of size $\sim 2^{\varepsilon t}\cdot$shortest
  - $t=\omega(\log P)$ ➔ $2^{\varepsilon t}\cdot QR > \det(B)^{1/t+1}$
  - Contemporary lattice reduction not strong enough

Why this algorithm fails



Conclusions for Part I

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- **A Simple Scheme that supports computing low-degree polynomials on encrypted data**
  - Any fixed polynomial degree can be done
  - To get degree-d, ciphertext size must be $\omega(nd^2)$
- **Already can be used in applications**
  - E.g., the keyword-match example
- **Next we turn it into a fully-homomorphic scheme**

[redacted]

Part II

Fully Homomorphic Encryption

Bootstrapping [Gentry 09]

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- So far, can evaluate low-degree polynomials

x_1

x_2

…

x_t

$f(x_1, x_2, \ldots, x_t)$

Bootstrapping [Gentry 09]

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- So far, can evaluate low-degree polynomials

x_1
x_2
…
x_t

$f(x_1, x_2, \dots, x_t)$

- Can eval $y=f(x_1,x_2\dots,x_n)$ when x_i's are "fresh"
- But y is "evaluated ciphertext"
  - Can still be decrypted
  - But eval $Q(y)$ has too much noise

Bootstrapping [Gentry 09]

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- So far, can evaluate low-degree polynomials

x_1

x_2

…

x_t

$f(x_1, x_2, \ldots, x_t)$

- Bootstrapping to handle higher degrees:
- For a ciphertext c, consider $\mathrm{D}_c(sk) = \mathrm{Dec}_{sk}(c)$
  - Hope: $\mathrm{D}_c(*)$ has a low degree in sk
  - Then so are

$$A_{c_1,c_2}(sk) = \mathrm{Dec}_{sk}(c_1) + \mathrm{Dec}_{sk}(c_2)$$

and $$M_{c_1,c_2}(sk) = \mathrm{Dec}_{sk}(c_1) \times \mathrm{Dec}_{sk}(c_2)$$

Bootstrapping [Gentry 09]

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- Include in the public key also $\mathrm{Enc}_{pk}(sk)$

x_1 x_2

c_1 c_2

sk_1
sk_2
...
sk_n

Requires "circular security"

c

$\mathrm{M}_{c_1,c_2}(sk)$

$= \mathrm{Dec}_{sk}(c_1) \times \mathrm{Dec}_{sk}(c_2) = x_1 \times x_2$

- Homomorphic computation applied only to the "fresh" encryption of *sk*

Bootstrapping [Gentry 09]

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- Fix a scheme (Gen, Enc, Dec, Eval)
- For a class F of functions , denote
  - $C_F = \{ \text{Eval}(f, c_1,\ldots,c_t) : f \in F, c_i \in \text{Enc}(0/1) \}$
  - Encrypt some t bits and evaluate on them some $f \in F$
- Scheme *bootstrappable* if exists F for which:
  - Eval "works" for F
    - $\forall f \in F, c_i \in \text{Enc}(x_i), \text{Dec}(\text{Eval}(f,c_1,\ldots,c_t)) = f(x_1,\ldots,x_t)$
  - Decryption + add/mult in F
    - $\forall c_1,c_2 \in C_F, A_{c_1,c_2}(sk), M_{c_1,c_2}(sk) \in F$

Thm: Circular secure
& Boostrappable
➔ Homomorphic for any func.

Is our SHE Bootstrappable?

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- $\mathrm{Dec}_p(c) = \mathrm{LSB}(c) \oplus \mathrm{LSB}([[c/p]])$
  - We have $|c| \sim n^5$, $|p| \sim n^2$

c/p, rounded to nearest integer

- Naïvely computing $[[c/p]]$ takes degree $> n^5$
- Our scheme only supports degree ~ n
- Need to "squash the decryption circuit" in order to get a bootstrappable scheme
  - Similar techniques to [Gentry 09]

How to"Simplify" Decryption?

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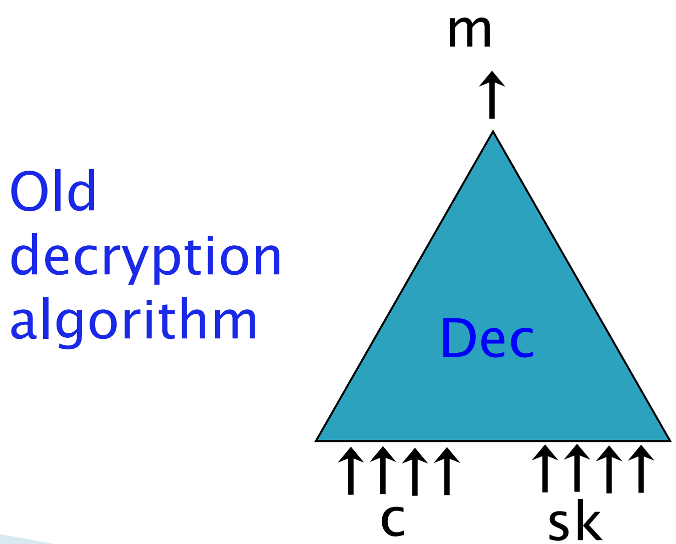
- Add to public key another "hint" about sk
  - Hint should not break secrecy of encryption
- With hint, ciphertext can be publically post-processed, leaving less work for Dec
- Idea is used in server-aided cryptography.



How to "simplify" decryption?

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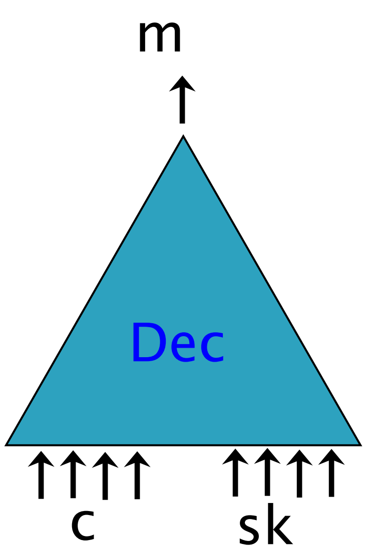
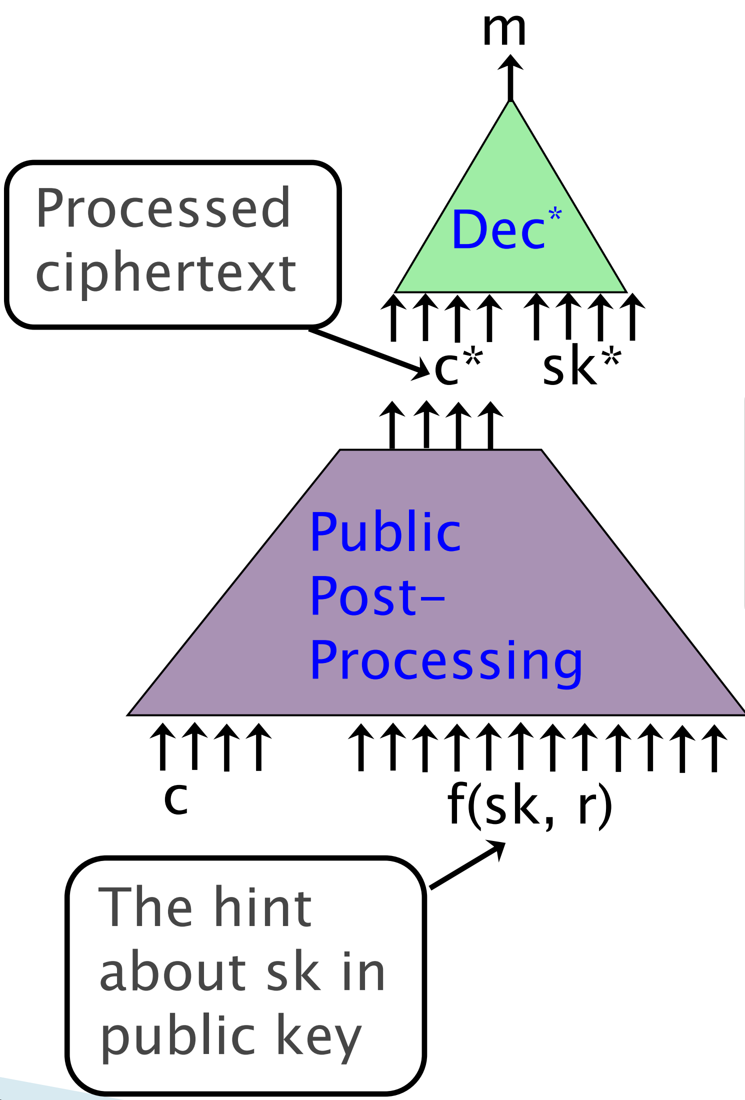
Old decryption algorithm



New approach



Hint in pub key lets anyone <u>post-process</u> the ciphertext, leaving less work for Dec*

The New Scheme

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- Old secret key is the integer p
- Add to public key many "real numbers"
  - $d_1, d_2, \ldots, d_t \in [0,2]$ (with precision of ~|c| bits)
  - $\exists$ **sparse** S for which $\Sigma_{i\in S}\, d_i = 1/p \bmod 2$
- Post Processing: $\psi_i = c \times d_i \bmod 2,\ i=1,\ldots,t$
  - New ciphertext is $c^* = (c, \psi_1, \psi_2, \ldots, \psi_i)$
- New secret key is char. vector of S $(\sigma_1, \ldots, \sigma_t)$
  - $\sigma_i = 1$ if $i \in S$, $\sigma_i = 0$ otherwise
  - $c/p = c \times (\Sigma\, \sigma_i d_i) = \Sigma\, \sigma_i \Psi_i \bmod 2$

$$Dec^*(c^*) = c - [[\Sigma_i\, \sigma_i \Psi_i]] \bmod 2$$

How to Add Numbers?

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- $\text{Dec}^*_\sigma(c^*) = \text{LSB}(c) \oplus \overbrace{\text{LSB}([[\ \Sigma_i \underbrace{\sigma_i \psi_i}_{a_i \in [0,2]}\ ]])}^{b \in \{0,1\}}$

$a_{1,0}$	$a_{1,-1}$	…	$a_{1,1-p}$	$a_{1,-p}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,1-p}$	$a_{2,-p}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,1-p}$	$a_{3,-p}$
…	…	…		…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,1-p}$	$a_{t,-p}$
b				

The a_i's in binary:
each $a_{i,j}$ is either σ_i or 0

- **Grade-school addition**
  - What is the degree of $b(\sigma_1,\ldots,\sigma_t)$?

Grade School Addition

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$c_{1,0}$	$c_{1,-1}$	…	$c_{1,1-p}$	
$a_{1,0}$	$a_{1,-1}$	…	$a_{1,1-p}$	$a_{1,-p}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,1-p}$	$a_{2,-p}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,1-p}$	$a_{3,-p}$
…	…	…		…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,1-p}$	$a_{t,-p}$
				b_{-p}

Carry Bits

Result Bit

$$c_{1,0}c_{1,-1} \ldots c_{1,1-p}\, b_{-p} = \text{HammingWeight}(\text{Colum}_{-p}) \bmod 2^{p+1}$$

Grade School Addition

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$c_{2,0}$	$c_{2,-1}$	…		
$c_{1,0}$	$c_{1,-1}$	…	$c_{1,1-p}$	
$a_{1,0}$	$a_{1,-1}$	…	$a_{1,1-p}$	$a_{1,-p}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,1-p}$	$a_{2,-p}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,1-p}$	$a_{3,-p}$
…	…	…		…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,1-p}$	$a_{t,-p}$
			b_{1-p}	b_{-p}

$c_{2,0}c_{2,-1} \ldots c_{2,2-p}\, b_{1-p}$
$= \text{HammingWeight}(\text{Column}_{1-p})$
$\bmod 2^p$

Grade School Addition

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$c_{p,0}$				
…	…			
$c_{2,0}$	$c_{2,-1}$	…		
$c_{1,0}$	$c_{1,-1}$	…	$c_{1,1-p}$	
$a_{1,0}$	$a_{1,-1}$	…	$a_{1,1-p}$	$a_{1,-p}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,1-p}$	$a_{2,-p}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,1-p}$	$a_{3,-p}$
…	…	…		…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,1-p}$	$a_{t,-p}$
	b_{-1}	…	b_{1-p}	b_{-p}

$c_{p,0}b_{-1}$ = HamWeight(Col_{-1}) mod 4

Grade School Addition

$c_{p,0}$				
…	…			
$c_{2,0}$	$c_{2,-1}$	…		
$c_{1,0}$	$c_{1,-1}$	…	$c_{1,1-p}$	
$a_{1,0}$	$a_{1,-1}$	…	$a_{1,1-p}$	$a_{1,-p}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,1-p}$	$a_{2,-p}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,1-p}$	$a_{3,-p}$
…	…	…		…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,1-p}$	$a_{t,-p}$
b	b_{-1}	…	b_{1-p}	b_{-p}

- Express $c_{i,j}$’s as polynomials in the $a_{i,j}$’s

Small Detour: Elementary Symmetric Polynomials

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- Binary Vector $x = (x_1, \ldots, x_u) \in \{0,1\}^u$
- $e_k(x)$ = deg-k elementary symmetric polynomial
  - Sum of all products of k bits (u-choose-k terms)
- Dynamic programming to evaluate in time O(ku)
  - $e_i(x_1 \ldots x_j) = e_{i-1}(x_1 \ldots x_{j-1}) x_i + e_i(x_1 \ldots x_{j-1})$ (for $i \le j$)

	Λ	x_1	x_1,x_2	…	$x_1 \ldots x_{u-1}$	$x_1 \ldots x_u$
e_0	1	1	1		1	1
e_1	0					
…				$e_i(x_1 \ldots x_j)$		
e_k	0					

The Hamming Weight

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<u>Thm</u>: For a vector $x = (x_1, \ldots, x_u) \in \{0,1\}^u$,
i'th bit of $W=HW(x)$ is $e_{2^i}(x)$ mod 2

- Observe $e_{2^i}(x) = (W \text{ choose } 2^i)$
- Need to show: i'th bit of $W=(W \text{ choose } 2^i) \bmod 2$

- **Say $2^k \leq W < 2^{k+1}$ (bit k is MSB of W), show:**
  - For $i<k$, $(W \text{ choose } 2^i)=(W-2^k \text{ choose } 2^i) \quad \bmod 2$
  - For $i=k$, $(W \text{ choose } 2^k)=(W-2^k \text{ choose } 2^k)+1 \bmod 2$
- **Then by induction over W**
  - Clearly holds for $W=0$
  - By above, if holds for $W-2^k$
    then holds also for W

The Hamming Weight

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- **Use identity** $\binom{W}{2^i} = \sum_{j=0}^{2^i} \binom{W-2^k}{j}\binom{2^k}{2^i-j}$ **(*)**
  - For r=0 or r=2^k we have (2^k choose r) = 1
  - For 0<r<2^k we have (2^k choose r) = 0 mod 2

Numerator has more powers of 2 than denominator

$$\binom{2^k}{r} = \boxed{\frac{2^k}{r}}\ \boxed{\frac{(2^k-1)}{(r-1)} \cdots \frac{(2^k-r+1)}{1}}$$

integer $= \binom{2^k-1}{r-1}$

- **i<k: The only nonzero term in (*) is j=2^i**
- **i=k: The only nonzero terms in (*) are j=0 and j=2^k**

Back to Grade School Addition

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$c_{4,0}$				
$c_{3,0}$	$c_{3,-1}$			
$c_{2,0}$	$c_{2,-1}$	$c_{2,-2}$		
$c_{1,0}$	$c_{1,-1}$	$c_{1,-2}$	$c_{1,-3}$	
$a_{1,0}$	$a_{1,-1}$	$a_{1,-2}$	$a_{1,-3}$	$a_{1,-4}$
$a_{2,0}$	$a_{2,-1}$	$a_{2,-2}$	$a_{2,-3}$	$a_{2,-4}$
…	…	…	…	…
$a_{t,0}$	$a_{t,-1}$	$a_{t,-2}$	$a_{t,-3}$	$a_{t,-4}$

Carry Bits

Input Bits

b

Goal:
compute the degree of the polynomial **b**($a_{i,j}$'s)

Back to Grade School Addition

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$e_{16}(\ldots)$	$e_8(\ldots)$	$e_4(\ldots)$	$e_2(\ldots)$	
deg=1	deg=1	deg=1	deg=1	deg=1
deg=1	deg=1	deg=1	deg=1	deg=1
…	…	…	…	…
deg=1	deg=1	deg=1	deg=1	deg=1

Back to Grade School Addition

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$e_8(\ldots)$	$e_4(\ldots)$	$e_2(\ldots)$		
deg=16	deg=8	deg=4	deg=2	
deg=1	deg=1	deg=1	deg=1	deg=1
deg=1	deg=1	deg=1	deg=1	deg=1
…	…	…	…	…
deg=1	deg=1	deg=1	deg=1	deg=1

Back to Grade School Addition

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$e_4(\ldots)$	$e_2(\ldots)$			
deg=9	deg=5	deg=3		
deg=16	deg=8	deg=4	deg=2	
deg=1	deg=1	deg=1	deg=1	deg=1
deg=1	deg=1	deg=1	deg=1	deg=1
…	…	…	…	…
deg=1	deg=1	deg=1	deg=1	deg=1

Back to Grade School Addition

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$e_2(\ldots)$				
deg=9	deg=7			
deg=9	deg=5	deg=3		
deg=16	deg=8	deg=4	deg=2	
deg=1	deg=1	deg=1	deg=1	deg=1
deg=1	deg=1	deg=1	deg=1	deg=1
…	…	…	…	…
deg=1	deg=1	deg=1	deg=1	deg=1

Back to Grade School Addition

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deg=15				
deg=9	deg=7			
deg=9	deg=5	deg=3		
deg=16	deg=8	deg=4	deg=2	
deg=1	deg=1	deg=1	deg=1	deg=1
deg=1	deg=1	deg=1	deg=1	deg=1
…	…	…	…	…
deg=1	deg=1	deg=1	deg=1	deg=1

deg($\boxed{b}$) = 16

<u>**Claim**</u>: with p bits of precision,
$\deg(\ b(a_{i,j})\ ) \leq 2^p$

Our Decryption Algorithm

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- $\text{Dec}^*_\sigma(c^*) = \text{LSB}(c) \oplus \overbrace{\text{LSB}([[\, \Sigma_i \underbrace{\sigma_i \psi_i}_{a_i \in [0,2]}]])}^{b \in \{0,1\}}$

$a_{1,0}$	$a_{1,-1}$	…	$a_{1,1-p}$	$a_{1,-p}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,1-p}$	$a_{2,-p}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,1-p}$	$a_{3,-p}$
…	…	…		…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,1-p}$	$a_{t,-p}$
$\boxed{b}$				

The a_i's in binary: each $a_{i,j}$ is either σ_i or 0

- **degree(b) = 2^p**
  - We can only handle degree ~ n
  - Need to work with low precision, p ~ log n

Lowering the Precision

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- **Parameters ensure "noise" < p/2**
  - For degree-2n polynomials with $< 2^{n^2}$ terms (say)
  - With $|r|=n$, need $|p|\sim 3n^2$
- **What if we want a somewhat smaller noise?**
  - Say that we want the noise to be $< p/2n$
  - Instead of $|p|\sim 3n^2$, set $|p|\sim 3n^2+\log n$
    - Makes essentially no difference

<u>Claim</u>: c has noise < p/2n
& sparse subset size ≤ n-1
➔ enough to keep precision of log n bits for the ψ_i's

Lowering the Precision

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<u>Claim</u>: $|S| \le n-1$ & c/p within $1/2n$ from integer
$\Rightarrow$ enough to keep $\log n$ bits for the ψ_i's

<u>Proof</u>: ϕ_i = rounding of ψ_i to $\log n$ bits

- $|\phi_i - \psi_i| \le 1/2n \Rightarrow \sigma_i\phi_i = \begin{cases} \sigma_i\Psi_i & \text{if } \sigma_i=0 \\ \sigma_i\Psi_i \pm 1/2n & \text{if } \sigma_i=1 \end{cases}$

$\Rightarrow |\Sigma\sigma_i\phi_i - \Sigma\sigma_i\Psi_i| \le |S|/2n \le (n-1)/2n$

- $\Sigma\sigma_i\Psi_i = c/p$, within $1/2n$ of an integer

$\Rightarrow \Sigma\sigma_i\phi_i$ within $1/2n+(n-1)/2n=1/2$ of the same integer

$\Rightarrow [[\Sigma\sigma_i\phi_i]] = [[\Sigma\sigma_i\Psi_i]]$ QED

Bootstrappable, at last

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- $\mathrm{Dec}^*_\sigma(c^*) = \mathrm{LSB}(c) \oplus \mathrm{LSB}([[\, \Sigma_i \underbrace{\sigma_i \phi_i}_{a_i \in [0,2]}]])$

$a_{1,0}$	$a_{1,-1}$	…	$a_{1,-\log n}$
$a_{2,0}$	$a_{2,-1}$	…	$a_{2,-\log n}$
$a_{3,0}$	$a_{3,-1}$	…	$a_{3,-\log n}$
…	…	…	…
$a_{t,0}$	$a_{t,-1}$	…	$a_{t,-\log n}$

b

The a_i's in binary: each $a_{i,j}$ is either σ_i or 0

- $\mathrm{degree}(\, \mathrm{Dec}^*_{c^*}(\sigma)\,) \le n$
  $\rightarrow \mathrm{degree}(\, M_{c_1^* c_2^*}(\sigma)\,) \le 2n$
- Our scheme can do this!!!

Putting Things Together

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- Add to public key $d_1, d_2, \ldots, d_t \in [0,2]$
  - $\exists$ sparse S for which $\Sigma_{i \in S}\, d_i = 1/p \bmod 2$
- New secret key is $(\sigma_1, \ldots, \sigma_t)$, char. vector of S
- Also add to public key $u_i = \mathrm{Enc}(\sigma_i)$, $i=1,2,\ldots,t$
- Hopefully, scheme remains secure
  - Security with d_i's relies on hardness of "sparse subset sum"
    - Same arguments of hardness as for the approximate-GCD problem
  - Security with u_i's relies on "circular security" (just praying, really)

Computing on Ciphertexts

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- **To "multiply" c_1, c_2** (both with noise < p/2n)
  - Evaluate $M_{c_1,c_2}(*)$ on the ciphertexts $u_1, u_2, \ldots, u_t$
  - This is a degree-2n polynomial
  - Result is new c, with noise <p/2n
  - Can keep computing on it
- **Same thing for "adding" c_1, c_2**
- **Can evaluate any function**

Ciphertext Distribution

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- **May want evaluated ciphertexts to have the same distribution as freshly encrypted ones**
  - Currently they have more noise
- **To do this, add n more bits to p**
  - "Raw evaluated ciphertext" have noise $< p/2^n$
- **After encryption/evaluation, add noise ~ p/2n**
  - Note: DOES NOT more noise to Enc(σ) in public key
- **Evaluated, fresh ciphertexts now have the same noise**
  - Can show that distributions are statistically close

Conclusions

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- Constructed a fully-homomorphic (public key) encryption scheme
- Underlying somewhat-homomorphic scheme relies on hardness of approximate-GCD
- Resulting scheme relies also on hardness of sparse-subset-sum and circular security
- Ciphertext size is ~ n^5 bits
- Public key has ~ n^{10} bits