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Session 8: Constructions for Specific Functions of Interest

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Correction

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- **Extending OT [IKNP]**
  - Is fully simulatable
  - Depends on a non-standard security assumption "correlation robust" functions
    - Modern hash families should have this property

  - Security against malicious adversaries is based on the "cut and choose" approach
    - Increases the overhead by a factor of s to reduce cheating probability to 2^{-s}.

How efficient is Yao's protocol?

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- Example: the millionaires problem – comparing two N bit numbers

- What's the overhead?
  - Circuit size is linear in N
  - N oblivious transfers

Other applications

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- Two parties. Two **large** data sets.
- Example applications
  - Computing the Max?
  - Mean?
  - Median?
  - Intersection?

How efficient is generic secure computation?

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- If the circuit is not too large then generic secure two-party computation is efficient
- AES (key and plaintext are known to Alice and Bob, respectively) [PSSW09]
  - About 33,000 gates
  - 7/60/1114 sec for semi-honest/covert/malicious
- If the circuit is large: we currently need ad-hoc solutions.

Secure Computation of the Median

G. Aggarwal, N. Mishra and B. Pinkas, *Secure Computation of the K'th-ranked Element,* Eurocrypt'04.

k^{th}-ranked element (e.g. median)

- **Inputs:**
  - Alice: S_A Bob: S_B Large sets of **unique** items ($\in$D).
- **Output:**
  - $x \in S_A \cup S_B$ s.t. x has **k-1** elements smaller than it.
- **The median**
  - $k = (|S_A| + |S_B|) / 2$
- **Motivation:**
  - Basic statistical analysis of distributed data.
  - E.g., histogram of salaries.

Some information is always revealed

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- The k^{th}-ranked element reveals some information.
- Suppose $S_A = x_1,\ldots,x_{1000}$ (sorted)
  - Median of $S_A \cup S_B = x_{400}$
- Party A now learns that S_B contains at least 200 elements smaller than x_{400}
- But she shouldn't learn more

Using a generic solution…

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- **The Problem:**
  - The size of a circuit for computing the k^{th} ranked element is at least linear in k.
  - For the median, k is in the same order as the size of the inputs.

  - Generic constructions using circuits [Yao,…] have communication complexity which is linear in the circuit size, and therefore in k.

An (insecure) two-party median protocol

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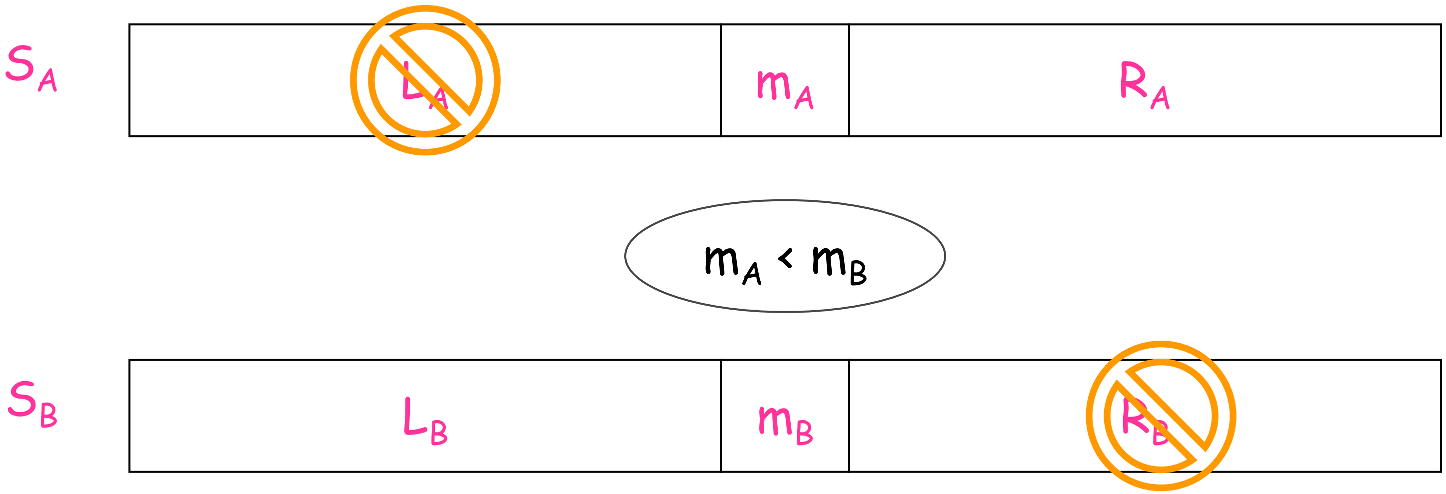

L_A lies below the median, R_B lies above the median. $|L_A| = |R_B|$

New median is same as original median!

Recursion → Need $\log n$ rounds
(assume each set contains $n=2^i$ items)

A Secure two-party median protocol

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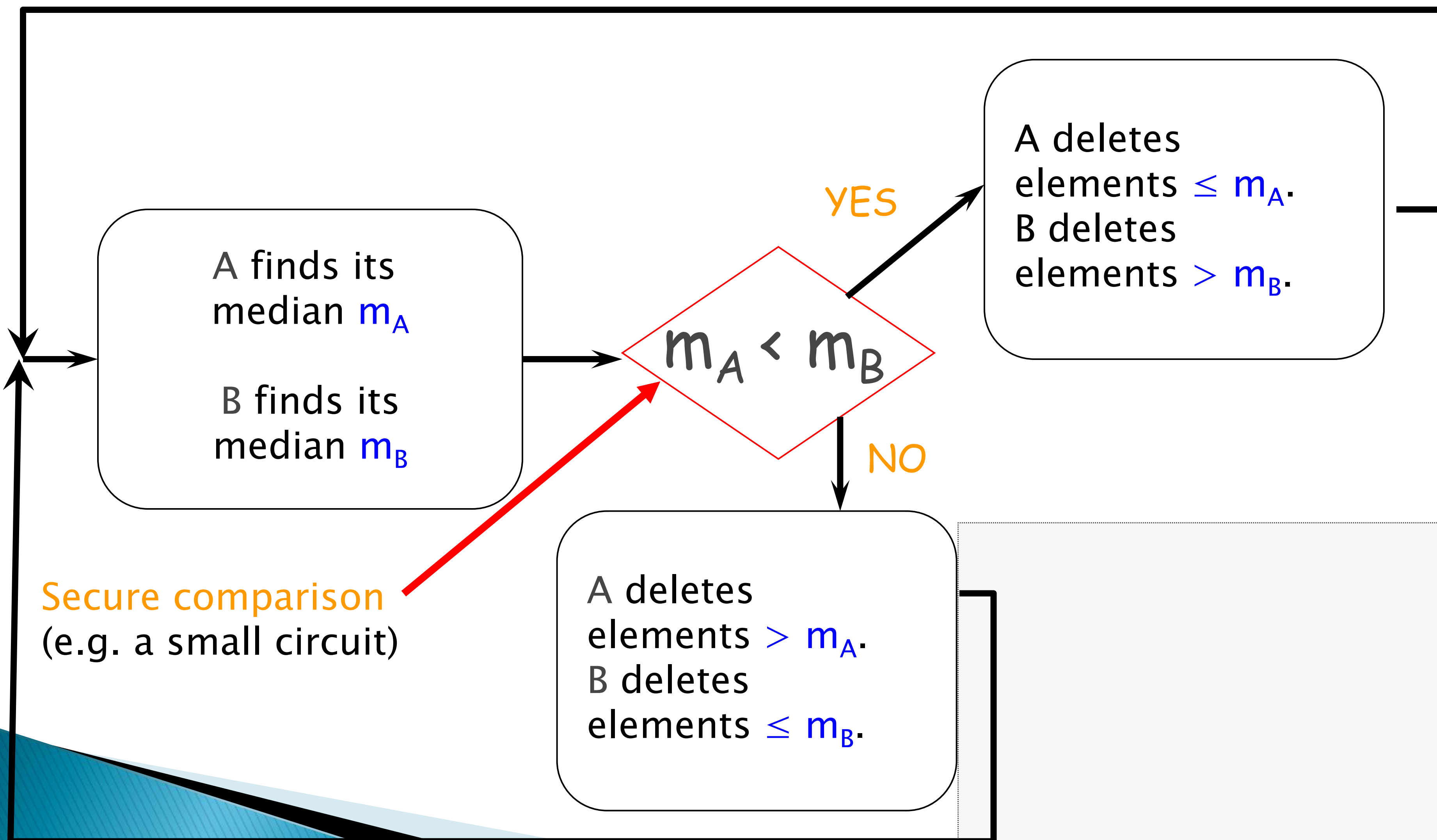

An example

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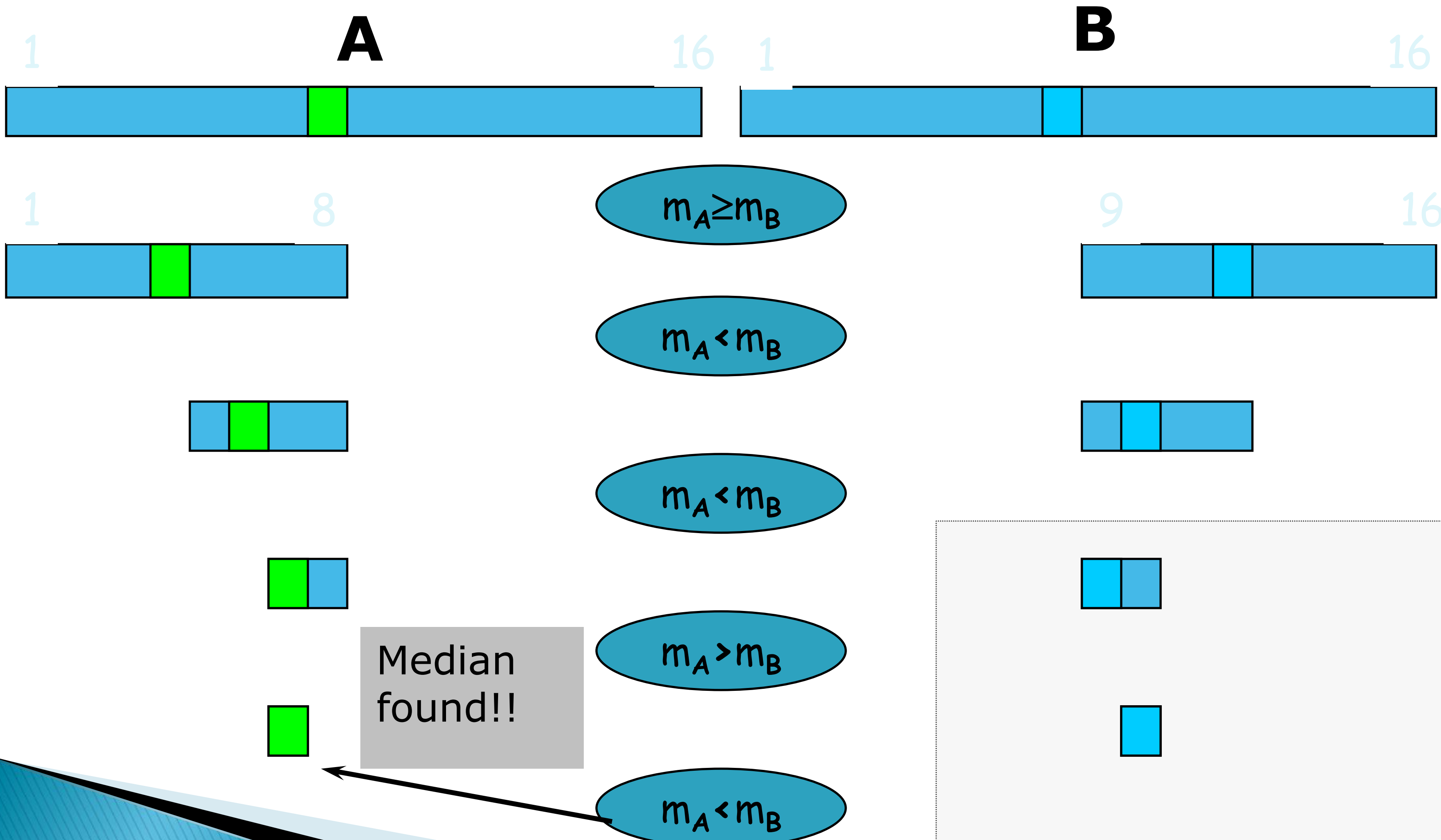

Proof of security

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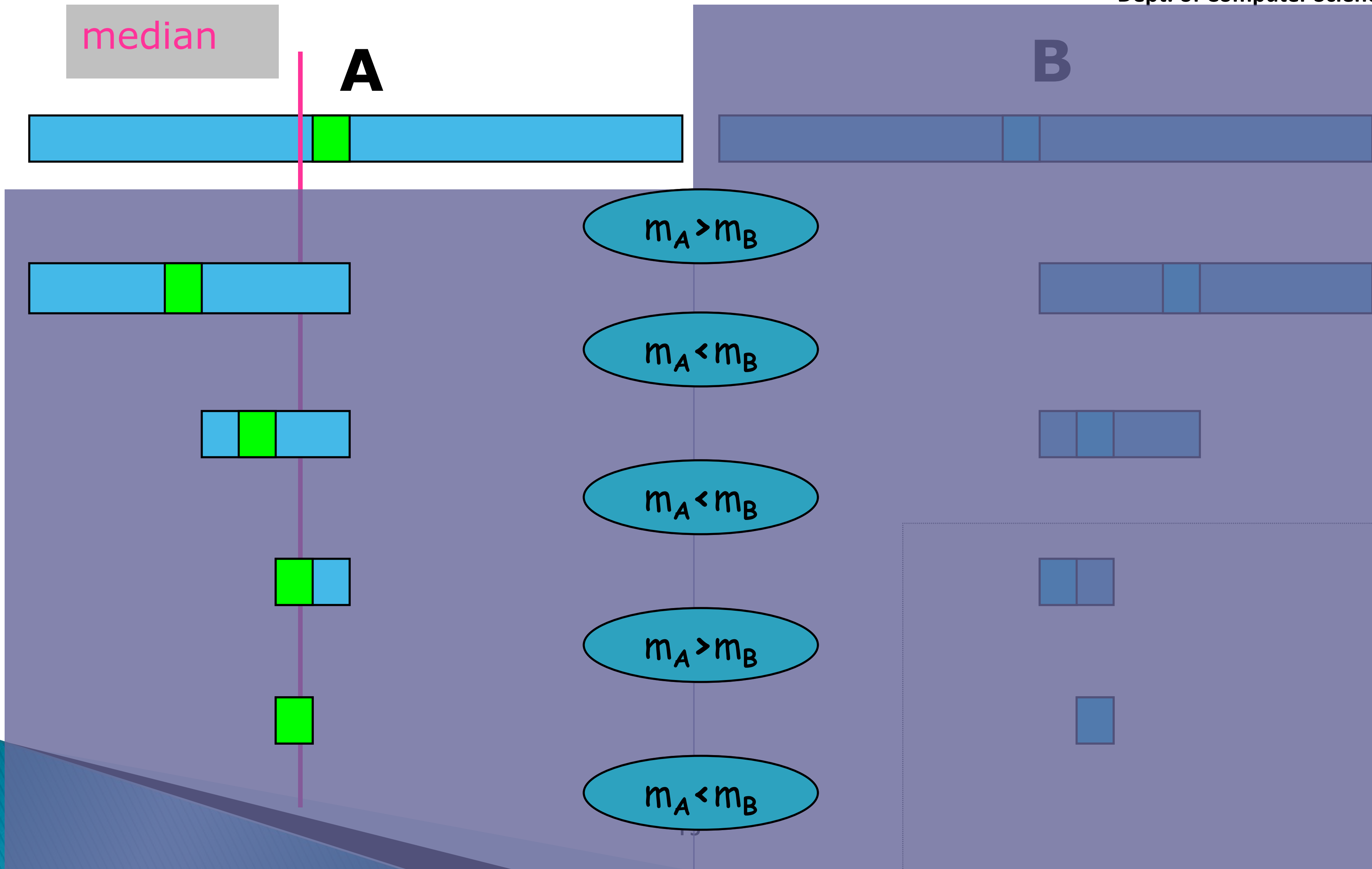

Proof of security

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- This is a proof of security for the case of semi-honest adversaries.

- **Security for malicious adversaries is more complex.**
    - The protocol must be changed to ensure that the parties' answers are consistent with some input.
    - Also, the comparison of the medians must be done by a protocol secure against malicious adversaries.

Arbitrary input size, arbitrary k

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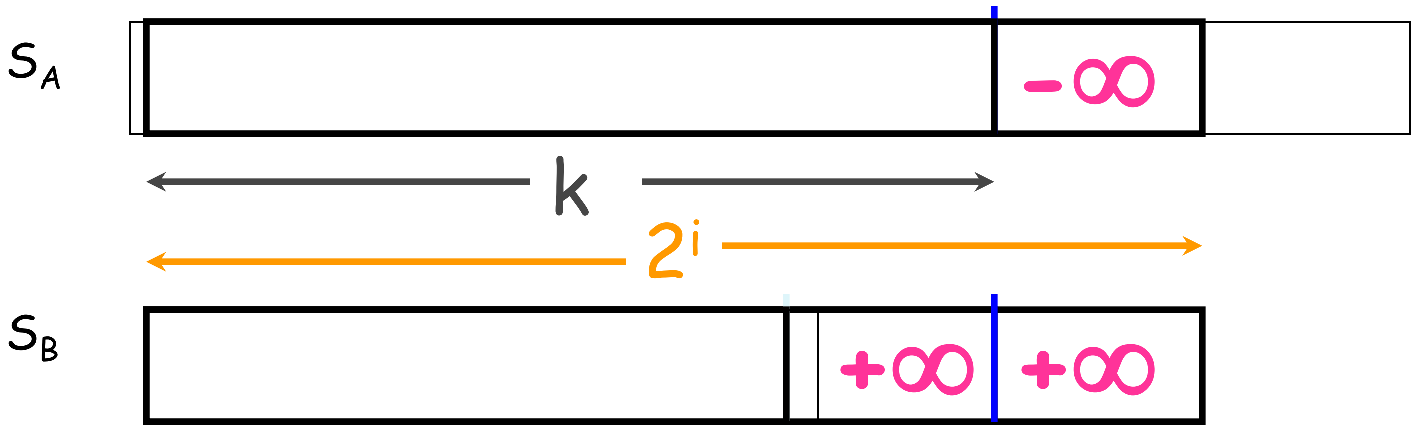

Now, compute the median of two sets of size k.

Size should be a power of 2.

median of new inputs = k^{th} element of original inputs

Hiding size of inputs

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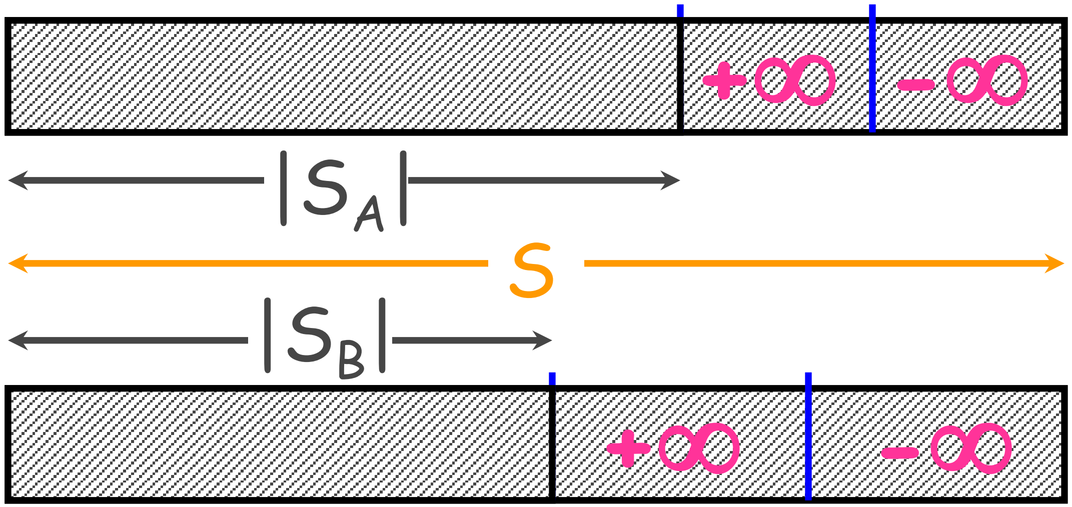
- Can search for k^{th} element without revealing size of input sets.
- However, $k=n/2$ (median) reveals input size.
- Solution: Let $S=2^i$ be a bound on input size.



Median of new datasets is same as median of original datasets

A Protocol secure against malicious adversaries

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- The parties can choose arbitrary inputs to the comparisons.
- For example,
  - In Step 1 claim that $m_A=100$, and be told that $m_A<m_B$ (therefore A must remove all items $\leq$mA).
  - In step 2 claim that $m_A=10$…
- We change the protocol so that even if input values are chosen adaptively during the protocol, they correspond to a valid input that can be sent to the TTP.

Protocol secure against malicious adversaries

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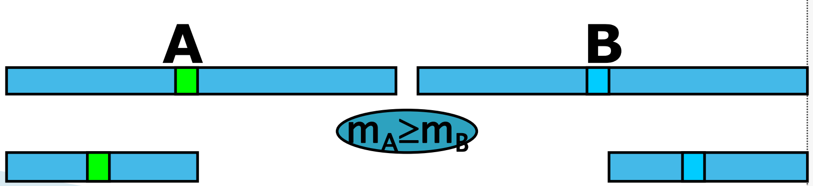
- **The modified protocol:**
  - Initialize bounds $L_A = L_B = -\infty$, $U_A = U_B = \infty$.
  - Each comparison protocol must be secure against malicious parties and verify that
    - $L_A < m_A < U_A$
    - $L_B < m_B < U_B$
  - If the verification succeeds, then
    - If $m_A \geq m_B$ then set $U_A = m_A$ and $L_B = m_B$
    - Otherwise set $L_A = m_A$ and $U_B = m_B$

The bounds ensure that m_A and m_B are consistent with previous inputs



Implementing the secure computation

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- The bounds L_A, L_B, U_A, U_B must not be revealed to any party, but rather be internal values of the secure computation.
- The secure computation is run in phases, where each phase must pass updated values of L_A, L_B, U_A, U_B to the next phase.
- Can be implemented using reactive computation
- Or, in a simpler way…

Implementing the secure computation

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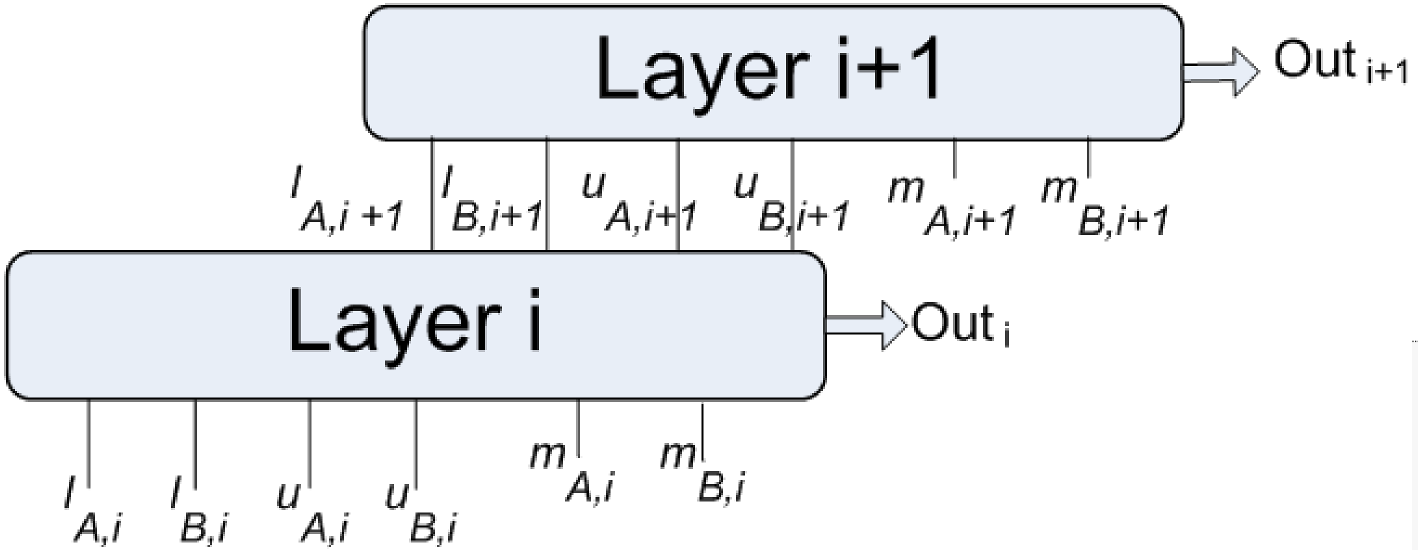
- The circuit is composed of layers.
- Each layer provides an external output, and has internal wires going into the next layer.



Implementing reactive computation

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- In each layer
  - provide A with
    - Shares of L_A, U_A, L_B, U_B
    - MACs of B's shares of these values
  - provide B with
    - Shares of L_A, U_A, L_B, U_B
    - MACs of A's shares of these values
- **In the next level**
  - A and B inputs these values.
  - The circuit checks the MACs, and reconstructs L_A, U_A, L_B, U_B from shares.

Proof of security

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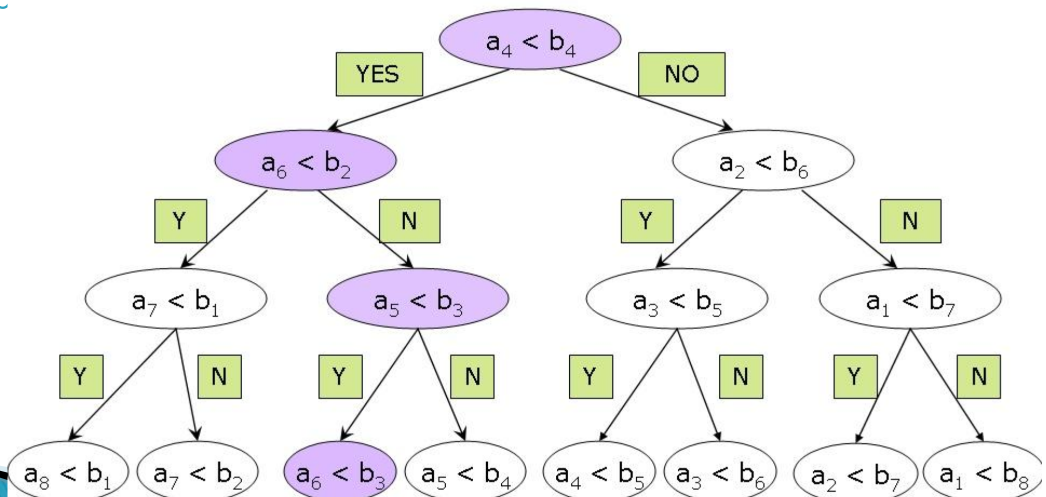
- Must show that for every adversary A’ in real model there is a simulator A’’ in the ideal model, etc…
- The operation of A’ in the real model can be visualized as following a path in a binary tree.

c



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Proof of security

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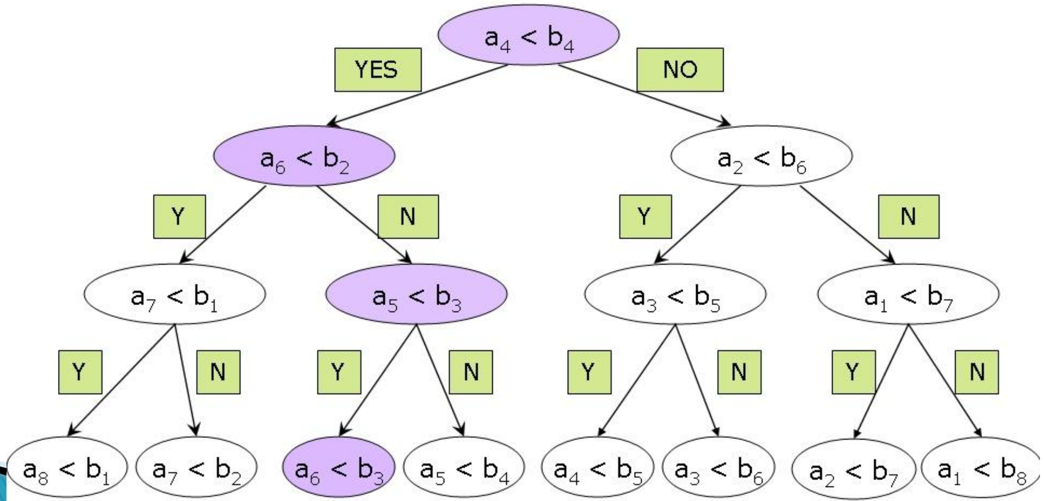
- If A' does not provide a legitimate input to a comparison (namely $m_A \notin (L_A,U_A)$) then the simulator aborts.
- Assume that the random input of A' is known to the simulator, and therefore A' is deterministic.



Proof of security

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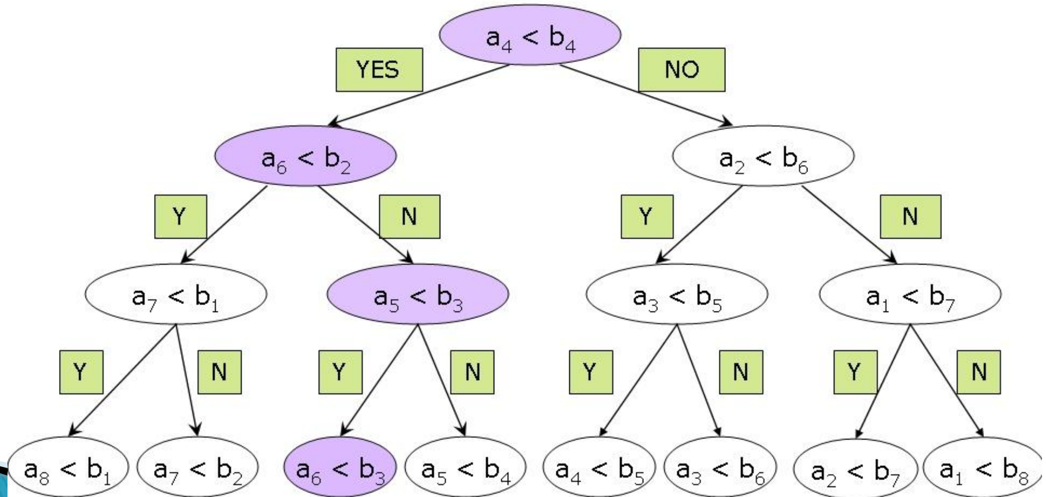
- The simulator runs the protocol with A', rewinding over all execution paths in the tree.
- Learns the inputs of A' to all comparisons. The inputs to the leaves correspond to the sorted input of A'.
- The simulator sends this input to TTP. Based on the result, it simulates the real execution path with A'.



The multi-party case

- Input: Party P_i has set S_i, i=1..n.
  (all values $\in$[a,b], where a and b are known)
- Output: k^{th} element of $S_1 \cup \ldots \cup S_n$
- Protocol (binary search): Set $m = (b-a)/2$. Repeat:
  - P_i uses the following input for a secure computation:
    L_i = # elements in S_i smaller than m.
    B_i = # times m appears in S_i.

  – The following is computed securely:
    - If $\Sigma L_i \geq k$, set $b=m$, $m=(m-a)/2$, else
    - If $\Sigma(L_i + B_i) \geq k$, stop. k^{th} element is m.
    - Otherwise, set $a=m$, $m = m+(b-m)/2$.

Conclusion

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- Efficient secure computation of the median.
  - Two-party: log k rounds * O(log D)
  - Multi-party: log D rounds * O(log D)
  - Very close to the communication complexity lower bound of log D bits.

- Malicious case is efficient too.
  - Do not use generic tools.
  - Instead, implement simple consistency checks.

Private matching and set intersection

M. Freedman, K. Nissim and B. Pinkas, *Efficient Private Matching and Set Intersection,* Eurocrypt'04.

The Scenario

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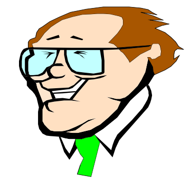
Client



Server

	Client	Server
Input:	$X = x_1 \ldots x_n$	$Y = y_1 \ldots y_n$
Output:	$X \cap Y$ only	nothing

- Shared interests (research, music)
- Credit rating
- Sharing intelligence between agencies (IARPA)
- Dating
- Genetic compatibility, etc

Implementation by a circuit?

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- Trivial circuit compares each (x_i,y_j) pair
  - $O(n^2)$ circuit size

- A more advanced circuit:
  - Sort the union of the two sets, using a sorting network.
  - If $x_i=y_j$ these two values will become adjacent.
  - Scan and search for identical adjacent values
  - O(nlogn) circuit size (with huge constant [AKS])

Basic tool: Additively homomorphic encryption

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- Public key encryption, such that
  - Given $E(x)$ it is possible to compute, without knowledge of the secret key, the value of $E(c \cdot x)$, for every c.
  - Given $E(x)$ and $E(y)$, it is possible to compute $E(x+y)$.
- We will use the notation
  - $E(x) \cdot E(y) = E(x+y)$
  - $E(x)^c = E(c \cdot x)$
- Applications
  - Voting
  - Many cryptographic protocols, such as keyword search, oblivious transfer…

Background on homomorphic encryption

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- “Standard” public key encryption schemes support Homomorphic operations with relation to multiplication
  - RSA
    - Public key: N, e. Private key: d.
    - $E(m) = m^e \bmod N$
    - $E(m_1) \cdot E(m_2) = E(m_1 \cdot m_2)$
  - El Gamal
    - Public key : p (or a similar group), $y=g^x$. Private key: x.
    - $E(m) = (g^r, y^r m)$
    - $E(m_1) \cdot E(m_2) = E(m_1 \cdot m_2)$

Background on additively homomorphic encryption

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- **Modified El Gamal**
  - $E(m) = (g^r, y^r g^m)$
  - $E(m_1) \cdot E(m_2) = (g^r, y^r g^{m_1 + m_2}) = E(m_1 + m_2)$
  - Decryption reveals $g^{m_1 + m_2}$
  - Computing m_1+m_2 is possible if m_1+m_2 is small

- **Paillier's cryptosystem**
  - Based on composite residuocity classes
  - Works in the group $Z^*_{n^2}$, where $n=pq$.
  - "Public-Key Cryptosystems Based on Composite Degree Residuosity Classes", Pascal Paillier, Eurocrypt'99.

The protocol (semi-honest case)

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- Client (C) defines a polynomial of degree n whose roots are her inputs $x_1,\ldots,x_n$

$$P(y) = (x_1-y)(x_2-y)\ldots(x_k-y) = a_0 + a_1 y + \ldots + a_k y^k$$

- C sends to server (S) homomorphic encryptions of polynomial's coefficients

$$Enc(a_0),\ldots , Enc(a_k)$$

The protocol

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- Note that
  - $Enc(P(y)) = Enc(a_0 + a_1 \cdot y^1 + \ldots + a_k \cdot y^k) =$
    $Enc(a_0) \cdot Enc(a_1)^{y} \cdot Enc(a_2)^{y^2} \cdot \ldots \cdot Enc(a_k)^{y^k}$
  - Therefore $\forall y$, server can compute $Enc(P(y))$

- The operation of the server
  - $\forall y_j$, choose random r_j and compute $Enc(r_j \cdot P(y_j) + y_j)$
  - This equals $Enc(y_j)$ if $y_j \in X$, and
    is random otherwise.
  - S sends (permuted) results back to C
  - C decrypts and learns $X \cap Y$

Variants of the basic protocol

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- The server computes $Enc(r_j \cdot P(y_j)+1)$
  - This equals $Enc(1)$ if $y_j \in X$, and is random otherwise.
  - The client decrypts, counts the number of 1's and learns $|X \cap Y|$.

- A different variant enables to compute whether $|intersection| > threshold$.

Security (semi-honest)

- **Client's privacy**
  - Server only sees semantically-secure enc's
  - We can simulate server's view by sending it enc's of arbitrary values.
- **Server's privacy**
  - Client can simulate her view in the protocol, given the output $X \cap Y$ alone:
    - Compute the enc's of items in $X \cap Y$ and of random items, and receive them in random order.

Efficiency

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- Communication is O(n)
  - ✓ C sends n coefficients
  - ✓ S sends n evaluations of polynomial
- Computation
  - ✓ Client encrypts and decrypts n values
  - ✘ Server:
    - $\forall y \in Y$, computes $Enc(r \cdot P(y)+y)$, using n exponentiations
    - Total of **$O(n^2)$** exponentiations ☹

Improving Efficiency (1)

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- Inputs typically from a "small" domain of D values. Represented by logD bits (say, 20)
- Use Horner's rule to compute polynomial:
  - $P(y) = a_0 + y\,(a_1 + \ldots y\,(a_{n-1} + y a_n)\ \ldots)$ instead of $P(y) = a_0 + a_1 y + a_{n-1} y^{n-1} + a_n y^n$
  - Now, exponents are only log D bits
  - Overhead of exponentiation is linear in |exponent|

→ Improvement by factor of |modulus|/log D, e.g., $1024/20 \approx 50$

Improving Efficiency (2): Hashing

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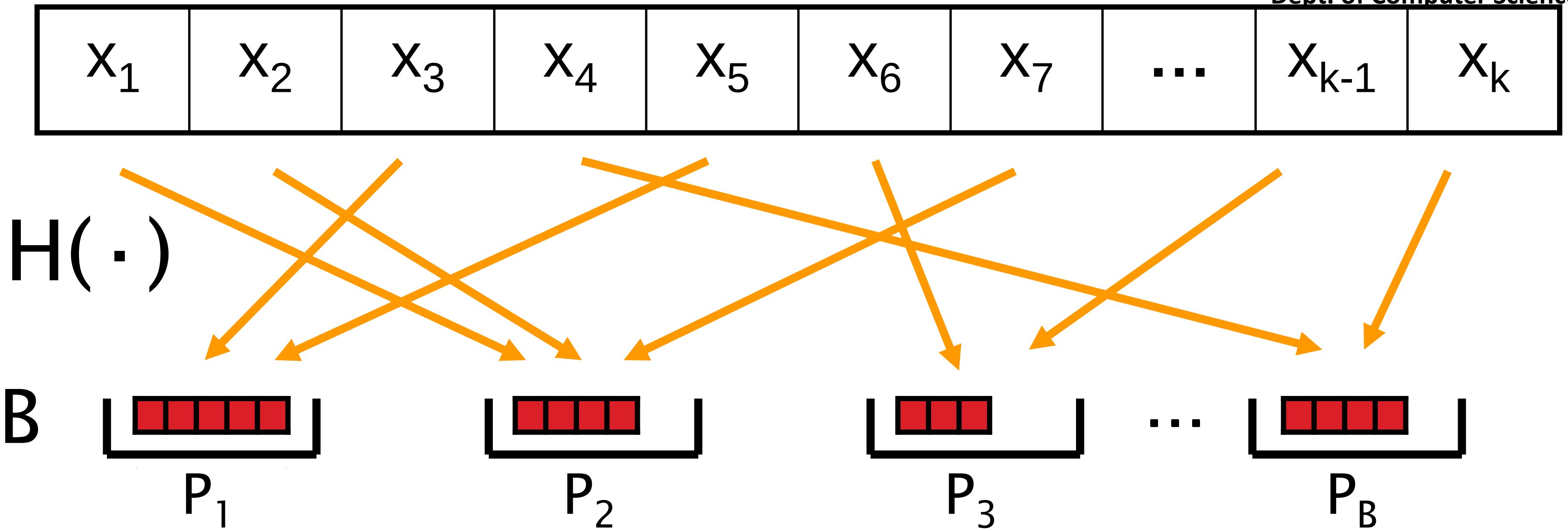

C uses $H(\cdot)$ to hash inputs to B bins (H indep. of inputs)
Let M bound max # of items in a bin.

Client defines B polynomials of deg M.

Each poly encodes x's mapped to its bin.

Improving Efficiency (2): Hashing

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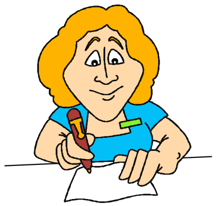

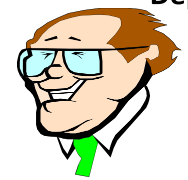

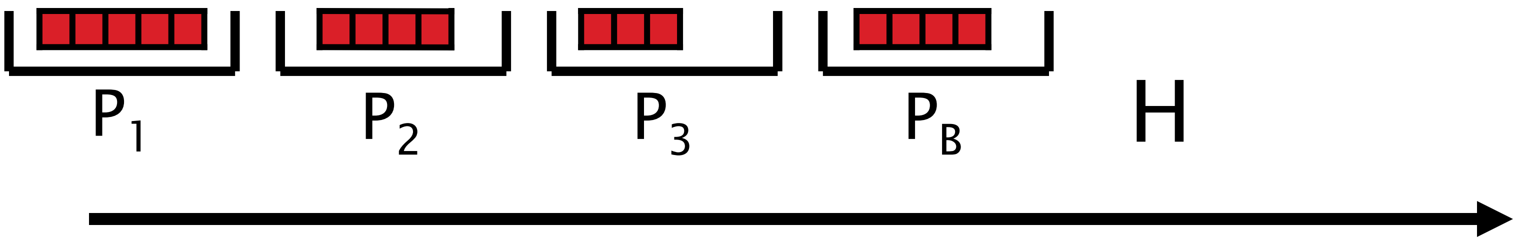

$$\forall y \in Y,\ i \leftarrow H(y),\ r \leftarrow \text{rand}$$

$$\text{Enc}(\ r \cdot P_i(y) + y\ )$$

- C sends B polynomials and H to server.
- For every y, S computes H(y) and evaluates the corresponding poly (of degree M)

Overhead with Hashing

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- Communication: $B \cdot M$
- Server: $n \cdot M$ short exp's, n full exp's
  ($P(y)$) ($r \cdot P_i(y) + y$)
- How large should M be?
- Simple hashing:
  - If the number of bins is $B=n$, then $M=O(\log n)$
  - Therefore
    - Communication $O(n\log n)$
    - Server computation $O(n\log n)$
  - Can do better…

Overhead with Hashing

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- Balanced allocations [ABKU]:
- $H=(h_1,h_2)$: Choose two bins, map item y to the less occupied bin among $h_1(y),h_2(y)$.
- It was shown that for $B = n/\ln \ln n$ the maximum bin size is $M=O(\ln \ln n)$
- Communication is $BM=O(n)$
- Server: $n \ln \ln n$ short exp, n full exp. (in practice $\ln \ln n \leq 5$)
- Client must check results in two bins

Overhead with Hashing

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- Cuckoo hashing [Pagh-Rodler]
- Map n items to $B \approx 2n$ bins of size $M=1$, or to a small stash (of size, say, 3).
- Each item y is found in either $h_1(y)$ or $h_2(y)$, or in the stash.
    - Details of the construction are omitted
- Communication, and server work are only O(n)

Actual run times

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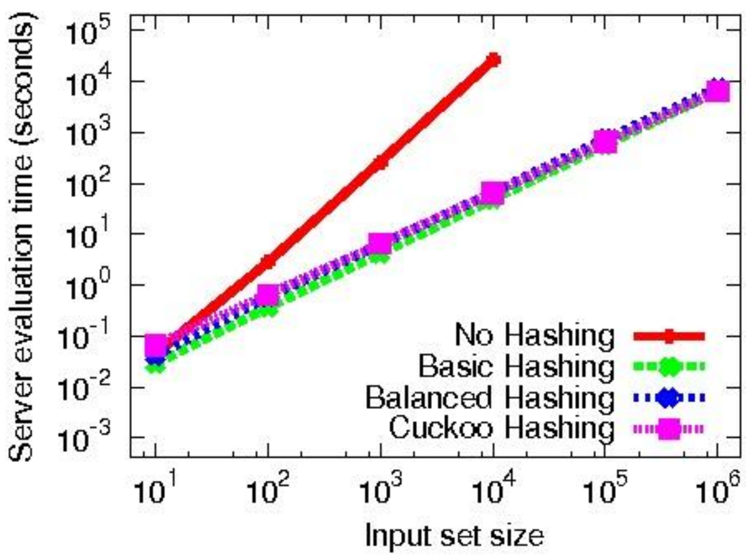
- Asymptotic run time of server with random hashing/balanced allocations/Cuckoo hashing is nlogn/nloglogn/n, respectively.
- For n=10,000, actual run times were 48/69/65 seconds, respectively.
- ????????



Actual run times – what happened?

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- Server computes $E(r \cdot P(y)+y)$
- The overhead of multiplying by r is independent of the degree. It is also a full exponentiation.
  - Experiments showed the overhead of evaluating $E(r \cdot P(y)+y)$ to be linear in d+6.5
- In the different methods S evaluates 1/2/3 polynomials, of degree logn/loglogn/O(1).
- Simple hashing is better since it evaluates a single polynomial.
- The other schemes are better only for larger values of n.