Winter School on Secure Computation and Efficiency
Bar-Ilan University, Israel 30/1/2011-1/2/2011

Bar-Ilan University
Dept. of Computer Science

Session 2: The Yao Construction and its Proof of Security

Yehuda Lindell
Bar-Ilan University

Yao's Protocol

Bar-Ilan University
Dept. of Computer Science

- **Protocol for general secure two-party computation**
  - Constant number of rounds
  - Secure for semi-honest adversaries
  - Many applications of the methodology beyond secure computation
- **General secure computation**
  - Can be used to securely compute any functionality
  - Based on the **Boolean circuit** for computing the function

Outline

Bar-Ilan University
Dept. of Computer Science

- **Garbled circuit**
  - An encrypted circuit together with a pair of keys (k_0,k_1) for every input wire so that given one key on every wire:
    - It is possible to compute the output (based on the input determined by the key provided on every wire)
    - It is not possible to learn anything else
- **Oblivious transfer**
  - Sender has x_0,x_1; receiver has b
  - Receiver obtains x_b only
  - Sender learns nothing

Outline

- **Yao's protocol**
  - Party P_1 constructs a **garbled circuit**
  - P_1 sends P_2 the keys associated with its input on its own input wires
    - P_1 sends only the keys so P_2 doesn't know what the actual input is
  - P_1 and P_2 use oblivious transfer so that for every one of P_2's input wires:
    - P_2 obtains the correct key associated with its input
    - P_1 learns nothing about P_2's input
  - P_2 computes the circuit and receives the output, and sends it back to P_1

Oblivious Transfer – Background

Bar-Ilan University
Dept. of Computer Science

- Trapdoor permutation (I,D,F,F^{-1})
  - I: samples a function **f** and trapdoor **t** in the family
  - D(f): uniformly samples a value in the domain of **f**
  - F(f,x): computes f(x)
  - $F^{-1}(t,y)$: computes $f^{-1}(y)$
  - Hard to invert a random **y**, given **f** (but not **t**)
- Enhanced trapdoor permutations
  - Hard to invert **y**, even given the random coins used to sample **y** (using D)

Oblivious Transfer – Background

Bar-Ilan University
Dept. of Computer Science

- Hard-core predicate B
  - Given y=f(x), can guess B(x) with probability only negligibly greater than ½
  - Equivalently, given y=f(x), the bit B(x) is pseudorandom

Oblivious Transfer Protocol

Bar-Ilan University
Dept. of Computer Science

- Sender's input: (z_0,z_1); receiver's input b
- Sender's first message:
  - Sender chooses (f,t) using sampling algorithm I
  - Sender sends f to receiver
- Receiver's first message:
  - Receiver chooses x_b and computes $y_b=f(x_b)$
  - Receiver chooses random y_{1-b}
  - Receiver sends (y_0,y_1) to sender
- Sender's second message:
  - Sender computes (x_0,x_1) by inverting
  - Sender computes $a_i = z_i \oplus B(x_i)$
  - Sender sends (a_0,a_1) to receiver
- Receiver outputs $z_b = a_b \oplus x_b$

Oblivious Transfer Protocol

Bar-Ilan University
Dept. of Computer Science

S (z_0,z_1) | **R** (b)

Choose (f,t) $\xrightarrow{\quad f \quad}$

Choose x_b, compute $y_b=f(x_b)$

Choose y_{1-b}

$\xleftarrow{\quad y_0,y_1 \quad}$

$x_0=f^{-1}(y_0)$

$a_0= z_0 \oplus B(x_0)$

$x_1=f^{-1}(y_1)$

$a_1= z_1 \oplus B(x_1)$

$\xrightarrow{\quad a_0,a_1 \quad}$ Output $z_b = a_b \oplus B(x_b)$

Security – P_1 Corrupted

Bar-Ilan University
Dept. of Computer Science

- Simulator is given (z_0,z_1); there is no output
  - SIM generates (f,t)
  - SIM chooses random y_0,y_1 using D(f)
  - SIM computes a_0,a_1 as in sender's instructions
- The transcript is exactly like a real protocol execution
  - Choosing x_b using D(f) and computing $y_b=f(x_b)$ is identical to choosing y_b using D(f)

Security – P_2 Corrupted

- Simulator is given (b,z_b)
  - SIM generates (f,t)
  - SIM chooses random x_b, y_{1-b} using D(f)
  - SIM computes $y_b = f(x_b)$
  - SIM computes $a_b = B(x_b) \oplus z_b$
  - SIM chooses a_{1-b} at random
- The transcript is indistinguishable from a real execution
  - By the hard-core property of B and the enhancement property of TDP, $B(x_{1-b})$ is indistinguishable from random

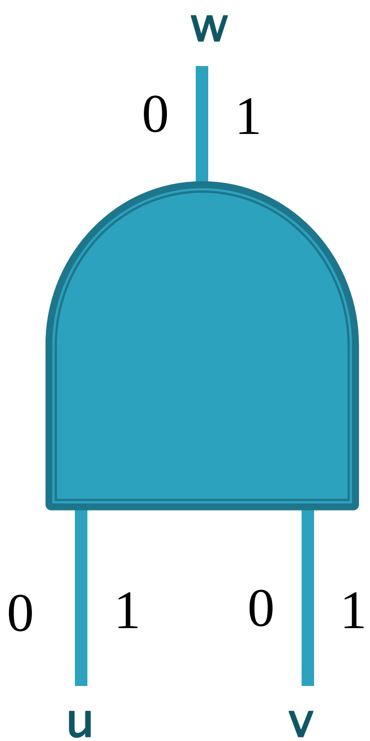
A Garbled Circuit

Bar-Ilan University
Dept. of Computer Science

- For the entire circuit, assign random values/keys to each wire (key k_0 for 0, key k_1 for 1)
- Encrypt each gate, so that given one key for each input wire, can compute the appropriate key on the output wire

An AND Gate

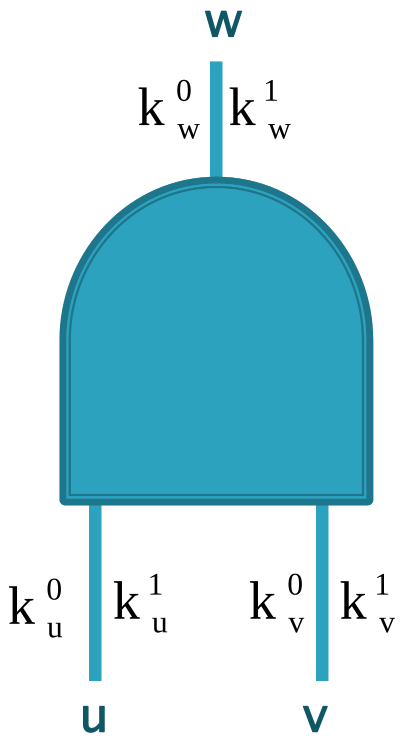
Bar-Ilan University
Dept. of Computer Science



u	v	w
0	0	0
0	1	0
1	0	0
1	1	1

An AND Gate with Garbled Values

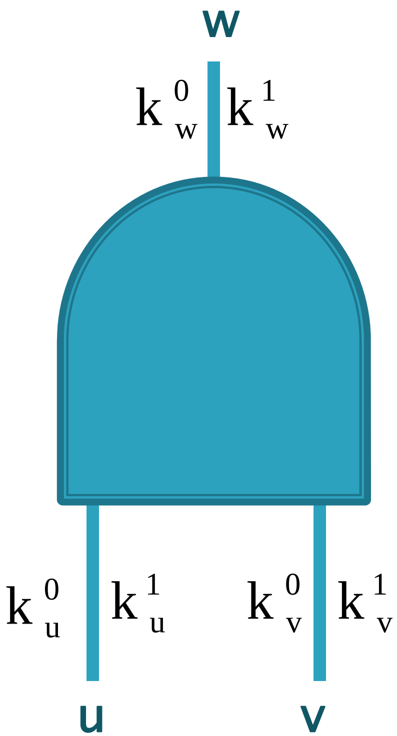
Bar-Ilan University
Dept. of Computer Science



u	v	w
k_u^0	k_v^0	k_w^0
k_u^0	k_v^1	k_w^0
k_u^1	k_v^0	k_w^0
k_u^1	k_v^1	k_w^1

A Garbled AND Gate

Tradition of Excellence
Bar-Ilan University
Dept. of Computer Science



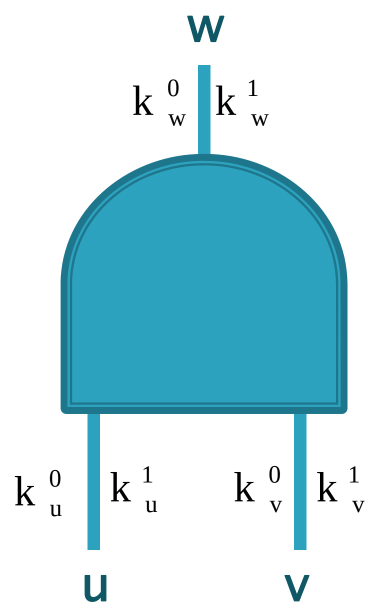
u	v	w
k_u^0	k_v^0	$E_{k_u^0}(E_{k_v^0}(k_w^0))$
k_u^0	k_v^1	$E_{k_u^0}(E_{k_v^1}(k_w^0))$
k_u^1	k_v^0	$E_{k_u^1}(E_{k_v^0}(k_w^0))$
k_u^1	k_v^1	$E_{k_u^1}(E_{k_v^1}(k_w^1))$

A Garbled AND Gate

Bar-Ilan University
Dept. of Computer Science

- The actual garbled gate

$$E_{k_u^1}(E_{k_v^0}(k_w^0))$$
$$E_{k_u^0}(E_{k_v^1}(k_w^0))$$
$$E_{k_u^1}(E_{k_v^1}(k_w^1))$$
$$E_{k_u^0}(E_{k_v^0}(k_w^0))$$



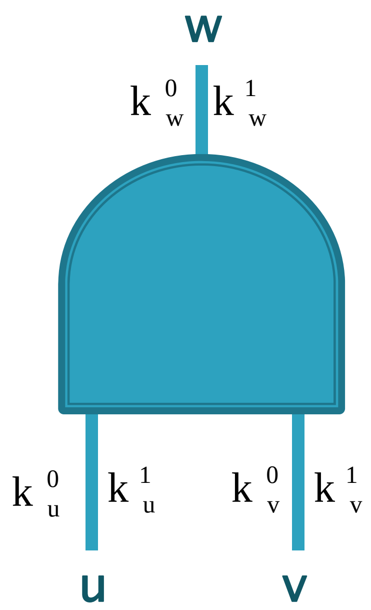
- Given k_u^0 and k_v^1 can obtain k_w^0 only
- Furthermore, since the table is permuted, the party has no idea if it obtained the 0 or 1 key

Output Translation

Bar-Ilan University
Dept. of Computer Science

- If the gate is an output gate, need to provide the "decryption" of the output wire
- Output translation table

$$\left[\left(0,\ k_w^0\right), \left(1,\ k_w^1\right)\right]$$



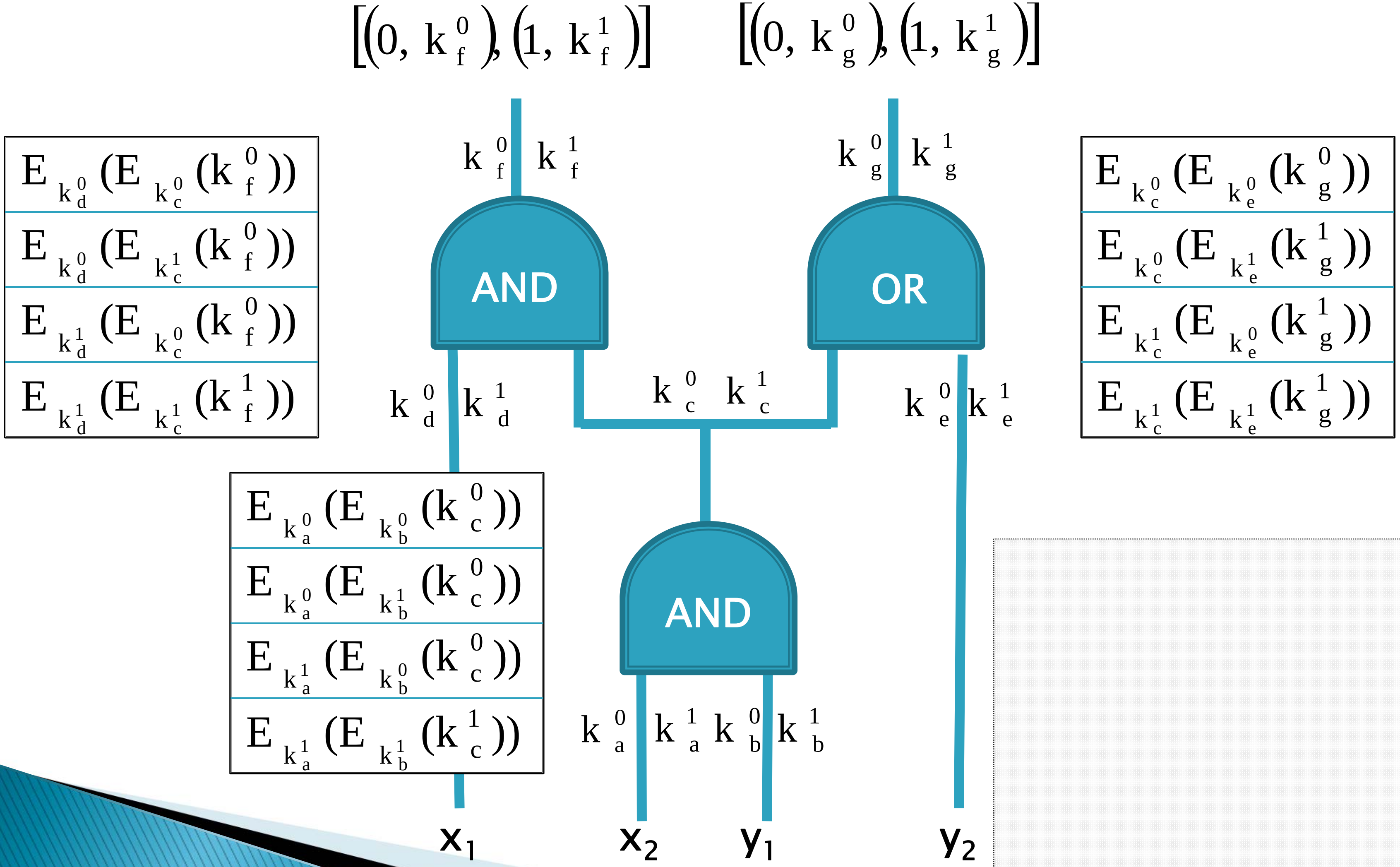
Constructing a Garbled Circuit

- **Given a Boolean circuit**
  - Assign garbled values to all wires
  - Construct garbled gates using the garbled values
- **Central property:**
  - Given a set of garbled values, one for each input wire, can compute the entire circuit, and obtain garbled values for the output wires
  - Given a translation table for the output wires, can obtain output
  - But, nothing but the output is learned!

An Example Circuit

(input wires $P_1 = d,a$; $P_2 = b,e$)

Bar-Ilan University
Dept. of Computer Science

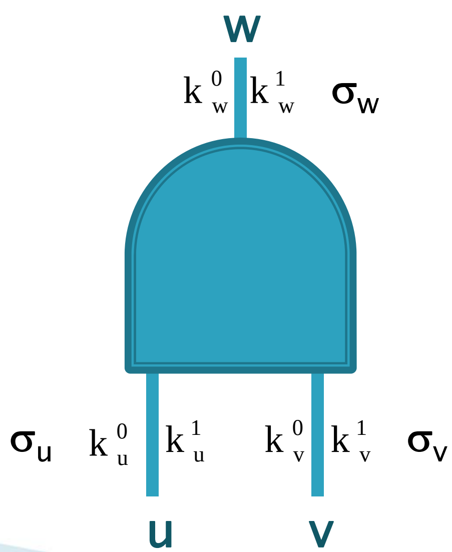


Computing a Garbled Circuit

- **How does the party computing the circuit know which is the "correct" entry**
  - It has one key on each wire, but symmetric encryption may decrypt "correctly" even with incorrect keys
- **Two possibilities** (actually many…)
  - Use encryption based on a PRF with redundant zeroes; only correct keys give redundant block
  - Add a bit to signal which ciphertext to decrypt

Computing a Garbled Circuit

- Option 1:
  - Encryption: $E_K(m) = [r , F_K(r) \oplus (m||0^n)]$
  - By pseudorandomness of F, probability of obtaining 0^n with an incorrect K is negligible
- Option 2:
  - For every wire, choose a random signal bit together with the keys

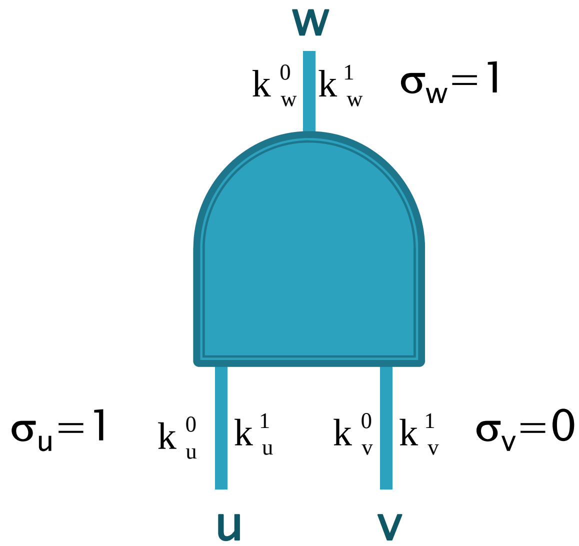


Computing a Garbled Circuit

Bar-Ilan University
Dept. of Computer Science

- The actual garbled gate

$$(0,0) \rightarrow E_{k_u^1}(E_{k_v^0}(k_w^0 \,\|\, 1))$$
$$(1,1) \rightarrow E_{k_u^0}(E_{k_v^1}(k_w^0 \,\|\, 1))$$
$$(0,1) \rightarrow E_{k_u^1}(E_{k_v^1}(k_w^1 \,\|\, 0))$$
$$(1,0) \rightarrow E_{k_u^0}(E_{k_v^0}(k_w^0 \,\|\, 1))$$



- Advantage
  - Computing the circuit requires just two decryptions per gate (rather than an average of 5)

Double-Encryption Security

Bar-Ilan University
Dept. of Computer Science

- Need to formally prove that given 4 encryptions of a garbled gate and only 2 keys
  - Nothing is learned beyond one output
- Actually, in order to simulate the protocol, we need something stronger
- Notation:
  - Double encryption: $\overline{E}(k_u, k_v, m) = E_{k_u}(E_{k_v}(m))$
  - Oracle: $\overline{E}(\cdot, k_v, \cdot)$

Double-Encryption Security

Bar-Ilan University
Dept. of Computer Science

$\mathsf{Expt}^{\mathsf{double}}_{\mathcal{A}}(n, \sigma)$

1. The adversary $\mathcal{A}$ is invoked upon input 1^n and outputs two keys k_0 and k_1 of length n and two triples of messages (x_0, y_0, z_0) and (x_1, y_1, z_1) where all messages are of the same length.
2. Two keys $k'_0, k'_1 \leftarrow G(1^n)$ are chosen for the encryption scheme.
3. $\mathcal{A}$ is given the challenge ciphertext $\langle \overline{E}(k_0, k'_1, x_\sigma), \overline{E}(k'_0, k_1, y_\sigma), \overline{E}(k'_0, k'_1, z_\sigma)\rangle$ as well as oracle access to $\overline{E}(\cdot, k'_1, \cdot)$ and $\overline{E}(k'_0, \cdot, \cdot)$
4. $\mathcal{A}$ outputs a bit b and this is taken as the output of the experiment.

Yao's Protocol

Bar-Ilan University
Dept. of Computer Science

- Input: x and y of length n
- P_1 generates a garbled circuit $G(C)$
  - k_L^0, k_L^1 are the keys on wire w_L
  - Let $w_1, \ldots, w_n$ be the input wires of P_1 and $w_{n+1}, \ldots, w_{2n}$ be the input wires of P_2
- P_1 sends P_2 the strings $k_1^{x1}, \ldots, k_n^{xn}$
- P_1 and P_2 run n OTs in parallel
  - P_1 inputs k_{n+i}^0, k_{n+i}^1
  - P_2 inputs y_i
- Given all keys, P_2 computes $G(C)$ and obtains $C(x,y)$
  - P_2 sends result to P_1

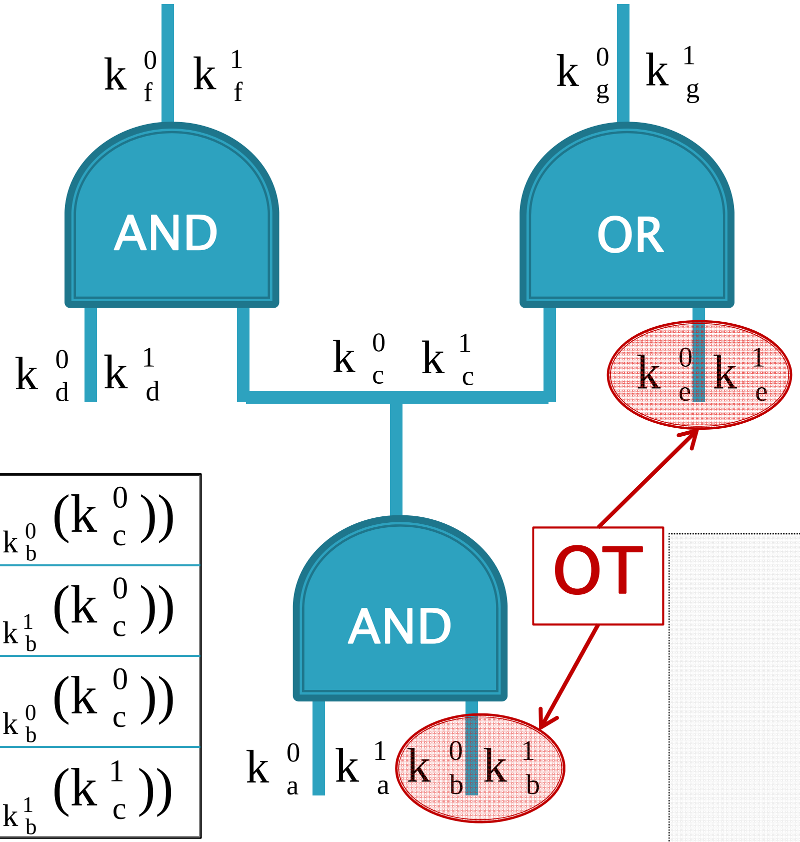
The Example Circuit

(input wires $P_1 = d,a$; $P_2 = b,e$)

Bar-Ilan University
Dept. of Computer Science

$$\left[\left(0, k_f^0\right), \left(1, k_f^1\right)\right] \qquad \left[\left(0, k_g^0\right), \left(1, k_g^1\right)\right]$$

$E_{k_d^0}(E_{k_c^0}(k_f^0))$
$E_{k_d^0}(E_{k_c^1}(k_f^0))$
$E_{k_d^1}(E_{k_c^0}(k_f^0))$
$E_{k_d^1}(E_{k_c^1}(k_f^1))$



$E_{k_c^0}(E_{k_e^0}(k_g^0))$
$E_{k_c^0}(E_{k_e^1}(k_g^1))$
$E_{k_c^1}(E_{k_e^0}(k_g^1))$
$E_{k_c^1}(E_{k_e^1}(k_g^1))$

$E_{k_a^0}(E_{k_b^0}(k_c^0))$
$E_{k_a^0}(E_{k_b^1}(k_c^0))$
$E_{k_a^1}(E_{k_b^0}(k_c^0))$
$E_{k_a^1}(E_{k_b^1}(k_c^1))$

Proof of Security – P_1 Corrupted

Bar-Ilan University
Dept. of Computer Science

- Party P_1's view consists only of the messages it receives in the oblivious transfers
- In the OT-hybrid model, P_1 receives no messages in the oblivious transfers
- Simulation:
  - Generate an empty transcript

Proof of Security – P_2 Corrupted

Bar-Ilan University
Dept. of Computer Science

- **More difficult case**
  - Need to construct a fake garbled circuit $G(C')$ that looks indistinguishable to $G(C)$
  - Simulated view contains keys to input wires and $G(C')$
  - $G(C')$ together with the keys computes to $f(x,y)$
  - Simulator doesn't know x, so cannot generate a real garbled circuit

Proof of Security – P_2 Corrupted

- **Simulator**
  - Given **y** and **z** = **f(x,y)**, construct a fake garbled circuit **G′(C)** that always outputs **z**
    - Do this by choosing wire keys as usual, but encrypting the **same output key** in all ciphertexts

$$E_{k_u^1}(E_{k_v^0}(k_w^0)) \qquad E_{k_u^1}(E_{k_v^1}(k_w^0))$$
$$E_{k_u^0}(E_{k_v^1}(k_w^0)) \qquad E_{k_u^0}(E_{k_v^0}(k_w^0))$$

    - This ensures that no matter the input, the same known garbled values on the output wires are received

Proof of Security – P_2 Corrupted

Bar-Ilan University
Dept. of Computer Science

- Simulator (continued)
  - Simulation of output translation tables
    - Let **k**,**k′** be the keys on the **i**[th] output wire; let **k** be the key encrypted in the preceding gate
    - If $\mathbf{z}_i = 0$, write [(0,k),(1,k′)]
    - If $\mathbf{z}_i = 1$, write [(0,k′),(1,k)]
  - Simulation of input keys phase
    - Input wires associated with P_1's input: send any one of the two keys on the wire
    - Input wires associated with P_2's input: simulate output of OT to be any one of the two keys on the wire

Proof of Security – P_2 Corrupted

Bar-Ilan University
Dept. of Computer Science

- Need to prove that the simulation is indistinguishable from the real
- First step – modify simulator as follows
  - Given **x** and **y** (just for the sake of the proof), label all keys on the wires as active or inactive
    - active: key is obtained on this wire upon inputs (**x**,**y**)
    - inactive: key is **not** obtained on wire upon inputs (**x**,**y**)
  - The single key to be encrypted in each gate is the active one
- This simulation is identical

Proof of Security – P_2 Corrupted

Bar-Ilan University
Dept. of Computer Science

- **Proven by a hybrid argument**
  - Consider a garbled circuit $G_L(C)$ for which:
    - The first L gates are generated as in the (alternative) simulation
    - The rest of the gates are generated honestly
- **Claim: $G_{L-1}(C)$ is indistinguishable from $G_L(C)$**
- **Proof:**
  - Difference is in L^{th} gate
  - **Intuition**: use indistinguishability of encryptions to say that cannot distinguish real garbled gate from one where same key is encrypted

Proof of Security – P_2 Corrupted

Bar-Ilan University
Dept. of Computer Science

- **Observation – L^{th} gate**
  - The encryption under both active keys is identical in both cases
  - The difference is what the inactive keys encrypt (only the next active key, or also the inactive)
    - The triple in the experiment are all encryptions under inactive keys
- **The problem**
  - The inactive keys in this gate may appear in other gates as well
    - Use oracles to generate rest…

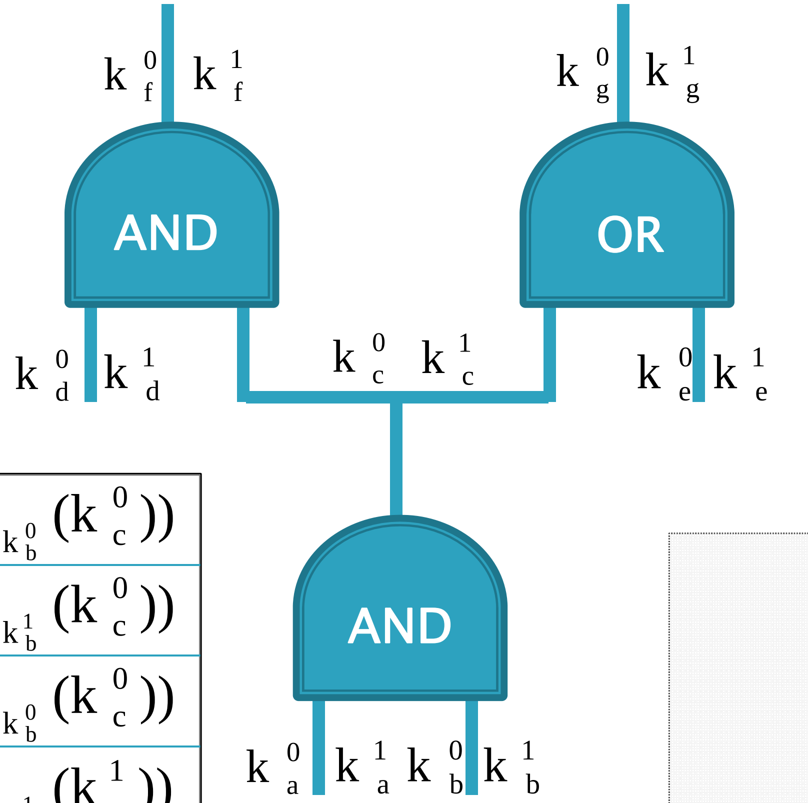
The Example Circuit

(input wires $P_1 = d,a$; $P_2 = b,e$)

Bar-Ilan University
Dept. of Computer Science

$[(0, k_f^0), (1, k_f^1)]$ $[(0, k_g^0), (1, k_g^1)]$

$E_{k_d^0}(E_{k_c^0}(k_f^0))$
$E_{k_d^0}(E_{k_c^1}(k_f^0))$
$E_{k_d^1}(E_{k_c^0}(k_f^0))$
$E_{k_d^1}(E_{k_c^1}(k_f^1))$



$E_{k_c^0}(E_{k_e^0}(k_g^0))$
$E_{k_c^0}(E_{k_e^1}(k_g^1))$
$E_{k_c^1}(E_{k_e^0}(k_g^1))$
$E_{k_c^1}(E_{k_e^1}(k_g^1))$

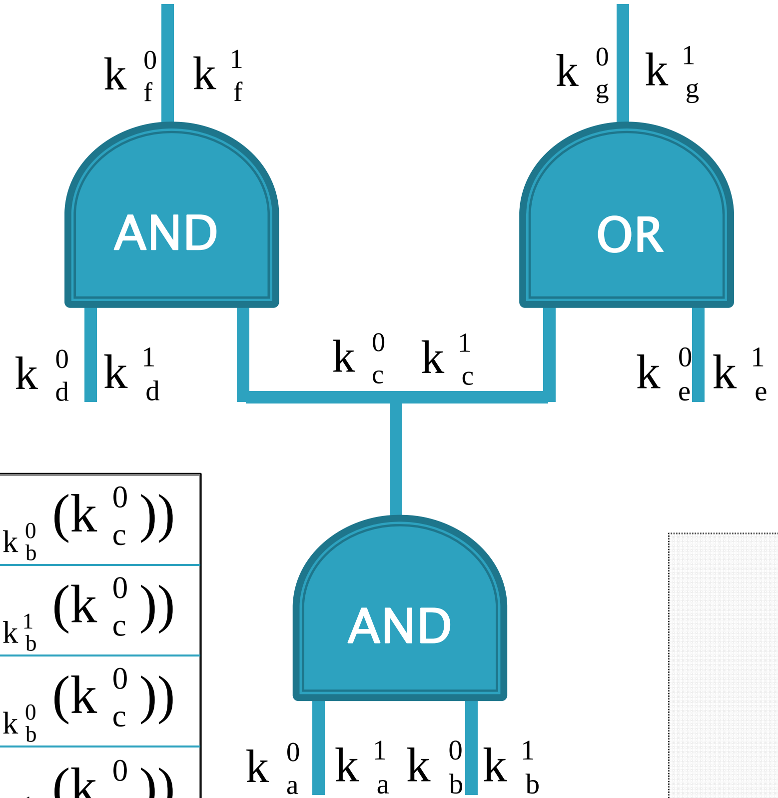
$E_{k_a^0}(E_{k_b^0}(k_c^0))$
$E_{k_a^0}(E_{k_b^1}(k_c^0))$
$E_{k_a^1}(E_{k_b^0}(k_c^0))$
$E_{k_a^1}(E_{k_b^1}(k_c^1))$

Simulator's Circuit (Output 01)

Bar-Ilan University
Dept. of Computer Science

$$\left[(0, k_f^0), (1, k_f^1)\right] \qquad \left[(0, k_g^0), (1, k_g^1)\right]$$

$E_{k_d^0}(E_{k_c^0}(k_f^0))$
$E_{k_d^0}(E_{k_c^1}(k_f^0))$
$E_{k_d^1}(E_{k_c^0}(k_f^0))$
$E_{k_d^1}(E_{k_c^1}(k_f^0))$

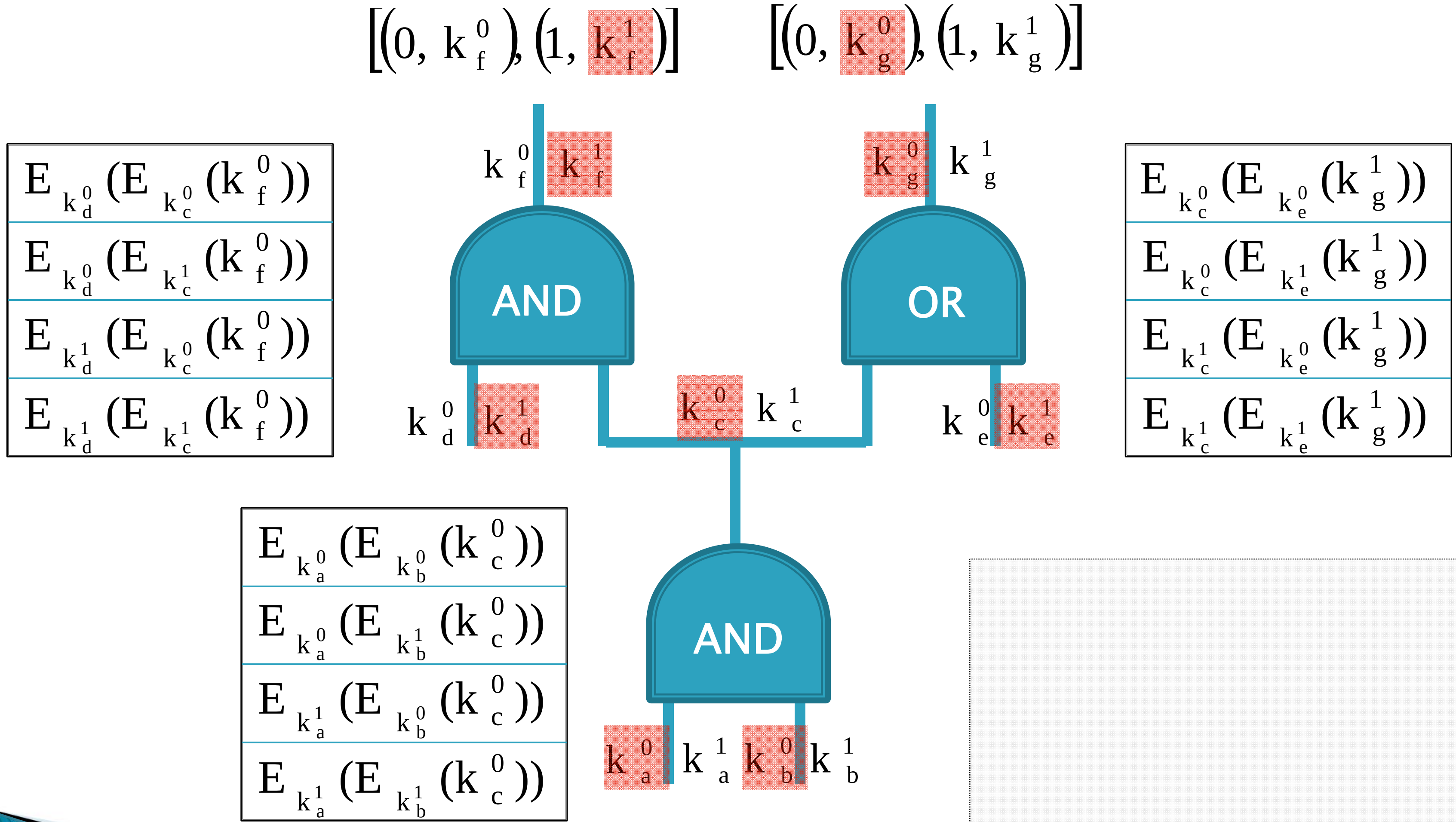


$E_{k_c^0}(E_{k_e^0}(k_g^1))$
$E_{k_c^0}(E_{k_e^1}(k_g^1))$
$E_{k_c^1}(E_{k_e^0}(k_g^1))$
$E_{k_c^1}(E_{k_e^1}(k_g^1))$

$E_{k_a^0}(E_{k_b^0}(k_c^0))$
$E_{k_a^0}(E_{k_b^1}(k_c^0))$
$E_{k_a^1}(E_{k_b^0}(k_c^0))$
$E_{k_a^1}(E_{k_b^1}(k_c^0))$

Inactive Keys

Input (da=01,be=10), Output (fg=01)

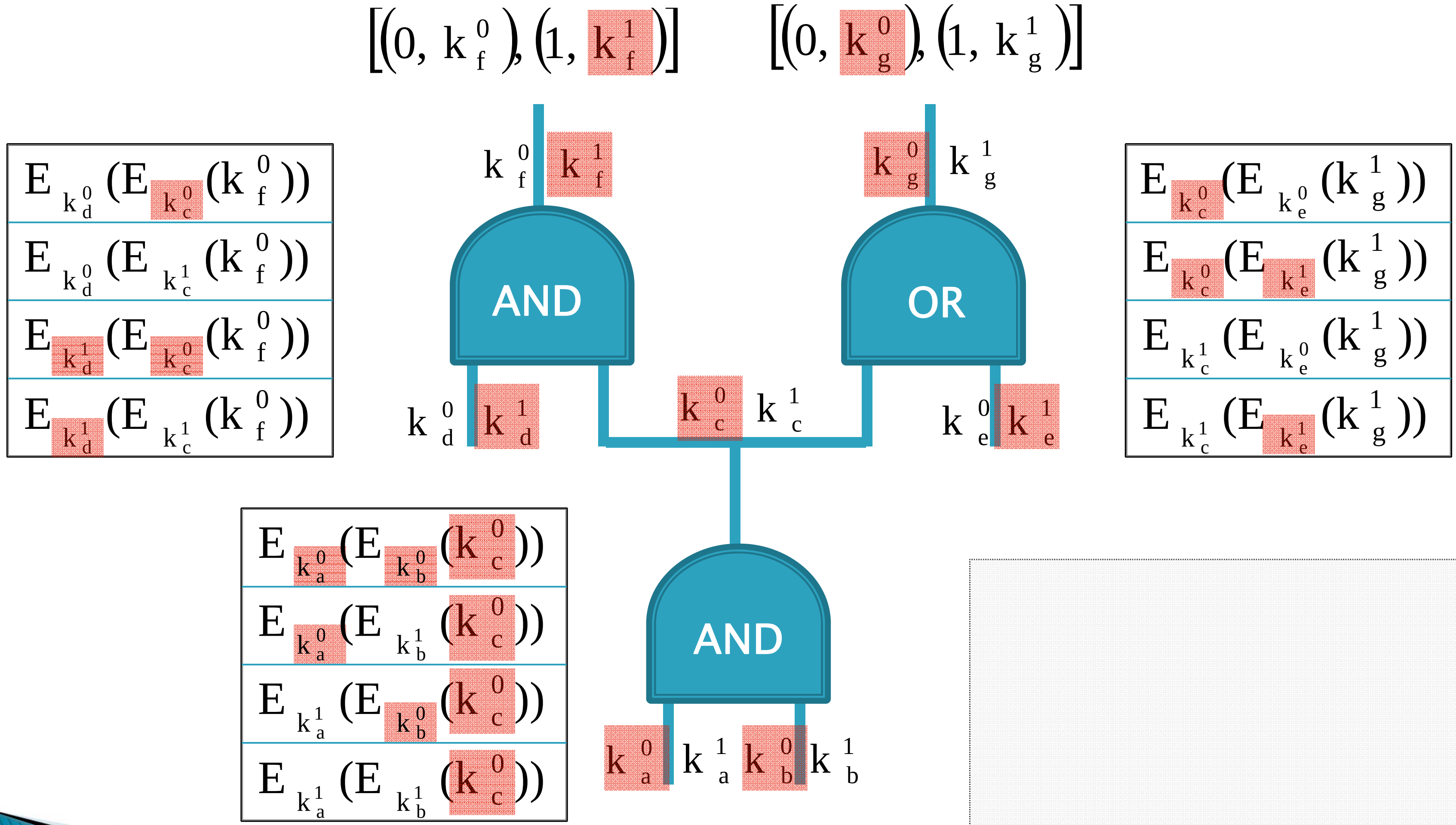


Bar-Ilan University
Dept. of Computer Science

Inactive Keys

Input (da=01,be=10), Output (fg=01)

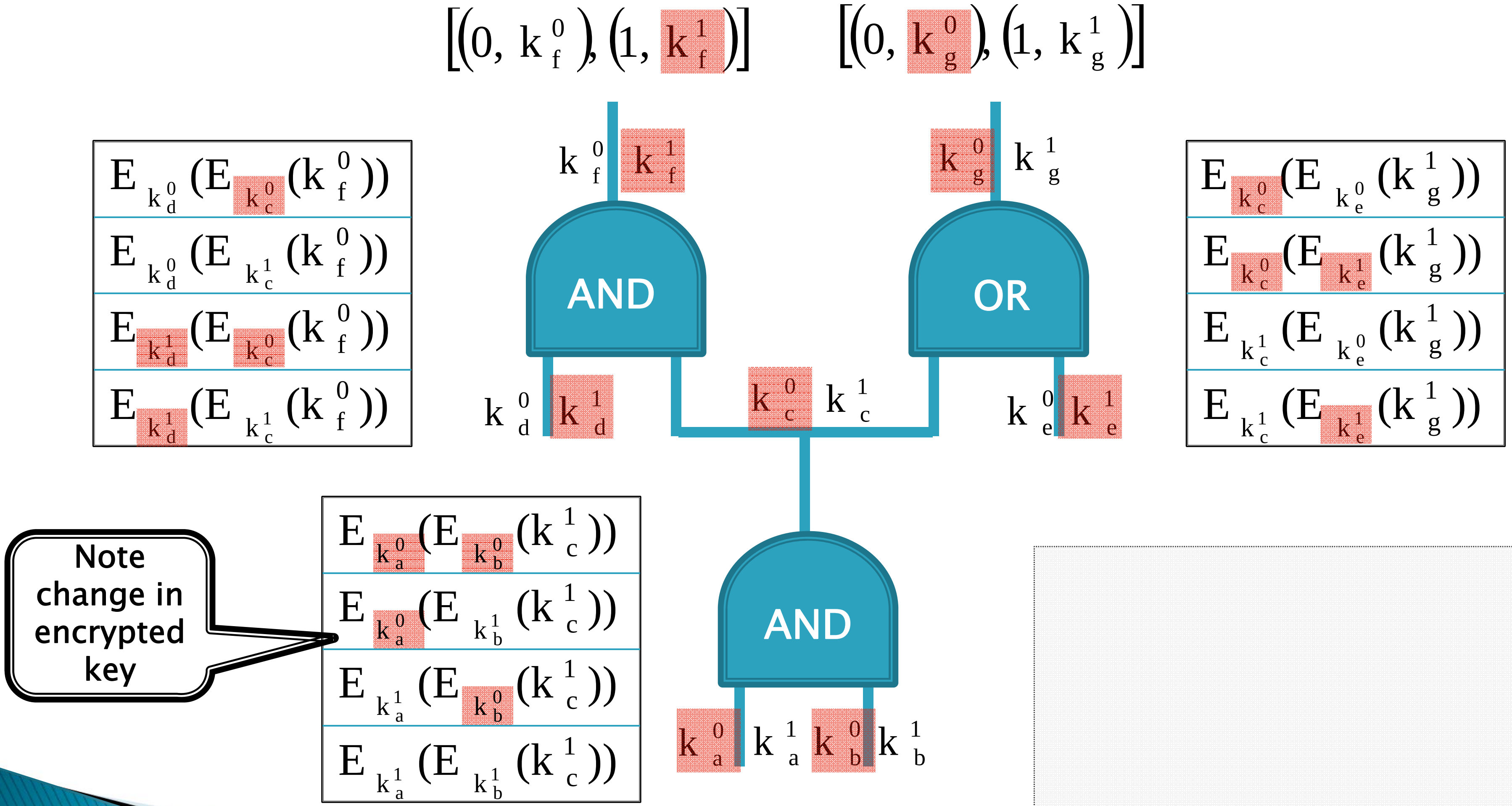
Bar-Ilan University
Dept. of Computer Science



Alternative Simulator

(Encrypt Active Keys Only)

Bar-Ilan University
Dept. of Computer Science



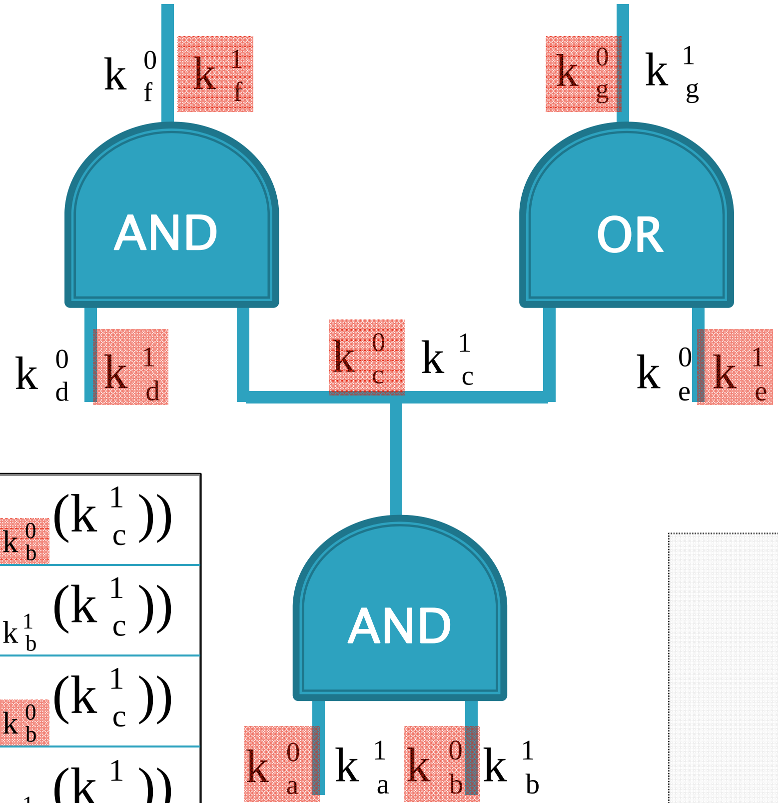
Hybrid on OR Gate – Simulated OR

Bar-Ilan University
Dept. of Computer Science

$[(0, k_f^0), (1, k_f^1)]$ $[(0, k_g^0), (1, k_g^1)]$

REAL

$E_{k_d^0}(E_{k_c^0}(k_f^0))$
$E_{k_d^0}(E_{k_c^1}(k_f^0))$
$E_{k_d^1}(E_{k_c^0}(k_f^0))$
$E_{k_d^1}(E_{k_c^1}(k_f^1))$



SIM

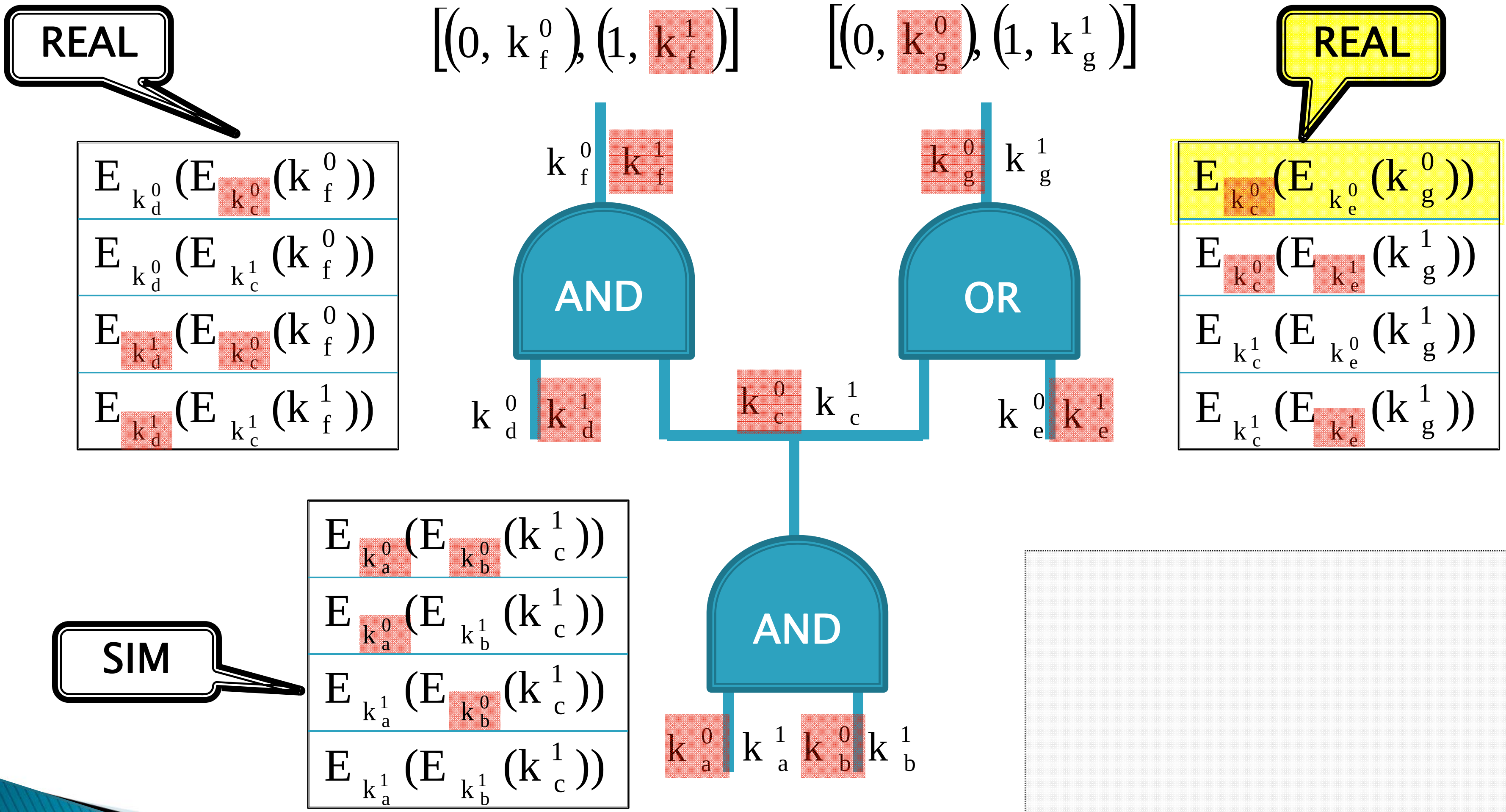
$E_{k_c^0}(E_{k_e^0}(k_g^1))$
$E_{k_c^0}(E_{k_e^1}(k_g^1))$
$E_{k_c^1}(E_{k_e^0}(k_g^1))$
$E_{k_c^1}(E_{k_e^1}(k_g^1))$

SIM

$E_{k_a^0}(E_{k_b^0}(k_c^1))$
$E_{k_a^0}(E_{k_b^1}(k_c^1))$
$E_{k_a^1}(E_{k_b^0}(k_c^1))$
$E_{k_a^1}(E_{k_b^1}(k_c^1))$

Hybrid on OR Gate – Real OR

Bar-Ilan University
Dept. of Computer Science



What's the Difference

Bar-Ilan University
Dept. of Computer Science

- In the simulated OR case, the inactive key k_c^0 encrypts the key k_g^1
- In the real OR case, the inactive key k_c^0 encrypts the key k_g^0
- Indistinguishability follows from the indistingushability of encryptions under the inactive key k_c^0

oving Indistinguishability

Tradition of Excellence
Bar-Ilan University
Dept. of Computer Science

Follows from the indistingushability of encryptions under the **inactive key** k_c^0

The good news

- Key k_c^0 is not encrypted anywhere (as data) because prior gates are simulated

The bad news

- The key k_c^0 needs to be used to construct the real AND gate for the hybrid

The solution

- The special double-encryption CPA game

ouble–Encryption Security

$^{\mathsf{double}}_{\mathcal{A}}(n, \sigma)$

The adversary $\mathcal{A}$ is invoked upon input 1^n and outputs two keys k_0 and k_1 of length n and two triples of messages (x_0, y_0, z_0) and (x_1, y_1, z_1) where all messages are of the same length.

Two keys $k'_0, k'_1 \leftarrow G(1^n)$ are chosen for the encryption scheme.

$\mathcal{A}$ is given the challenge ciphertext $\langle \overline{E}(k_0, k'_1, x_\sigma), \overline{E}(k'_0, k_1, y_\sigma), \overline{E}(k'_0, k'_1, z_\sigma) \rangle$ as well as oracle access to $\overline{E}(\cdot, k'_1, \cdot)$ and $\overline{E}(k'_0, \cdot, \cdot)$.[5]

$\mathcal{A}$ outputs a bit b and this is taken as the output of the experiment.

- k_0, k_1 (k_c^1, k_e^0) are active keys
- k'_0, k'_1 (k_c^0, k_e^1) are inactive keys
  - Can use oracle to generate the REAL AND gate

oof of Security – P_2 Corrupted

Bar-Ilan University
Dept. of Computer Science

Since each gate-replacement is ndistinguishable, using a hybrid argument we have that the distributions are ndistinguishable

QED

ficiency

Bar-Ilan University
Dept. of Computer Science

2-4 rounds (depending on OT and if both or one party receives output)

y| oblivious transfers

8|C| symmetric encryptions to generate circuit and 2|C| to compute it (using the signal bit)

For circuit of 33,000 gates:

- Between 7 and 14 seconds
- Between 503 and 3162 Kbytes
  (depends on encryption used)

alicious Adversaries

Bar-Ilan University
Dept. of Computer Science

Assume that the OT is secure for malicious adv:

- A corrupted P_1 cannot learn **anything** (it receives no messages in the protocol, in the hybrid-OT model)
  - Thus, we have privacy
- We can prove full security for the case of a corrupted P_2

This can be useful, but…

- Be warned that this doesn't compose with anything
- E.g., consider P_1 that builds circuit so that if P_2's first bit is 0, the circuit doesn't decrypt
  - If P_1 can detect this in the real world, privacy is lost

Bar-Ilan University
Dept. of Computer Science

ummary

- Can compute any functionality securely in presence of semi-honest adversaries

- Protocol is efficient enough for use, for circuits that are not too large

- Recommendation: read full proof