Pseudorandom Functions & Permutations

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Today's Plan

1. One-way functions and hardcore predicates
2. Pseudorandom generators
3. Pseudorandom functions and permutations
4. Symmetric encryption and MACs.

Reminder: Repeated Sampling From Pseudorandom Distributions

Claim 1

Let $G\colon \{0,1\}^n \mapsto \{0,1\}^{m(n)}$ be a pseudorandom generator and let $t \in \mathrm{poly}$, then $G^t\colon (\{0,1\}^n)^{t(n)} \mapsto (\{0,1\})^{t(n)}$, defined by

$$G^t(x_1,\ldots,x_{t(n)}) = G(x_1),\ldots,G(x_{t(n)})$$

is a pseudorandom generator.

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Proof: ? via hybrid

Part I

Pseudorandom Functions

Motivation Discussion

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Solution

Random Functions

Definition 2 (random functions)

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- The truth table of $\pi \stackrel{R}{\leftarrow} \Pi_n$ is a uniform string of length $2^n \cdot n$

Ensembles of Function Families

- $\mathcal{F} = \{\mathcal{F}_n\}_{n\in\mathbb{N}}$, where $\mathcal{F}_n = \{f\colon \{0,1\}^{m(n)} \mapsto \{0,1\}^{\ell(n)}\}$
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Efficient. $\exists$ poly-time algorithm that given $x \in \{0,1\}^{m(n)}$ and (a description of) $f \in \mathcal{F}_n$, outputs $f(x)$.

Pseudorandom Functions

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for any oracle-aided PPT D.

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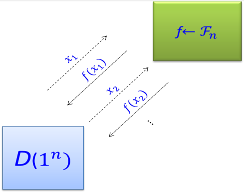

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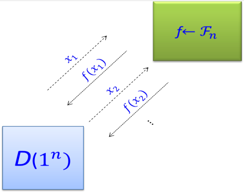

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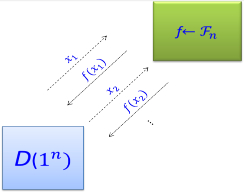

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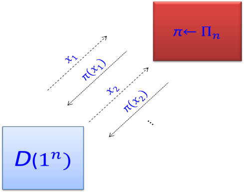

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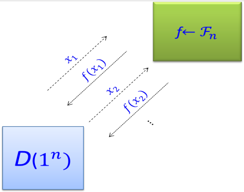

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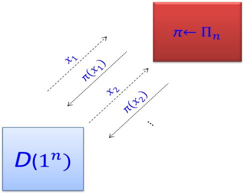

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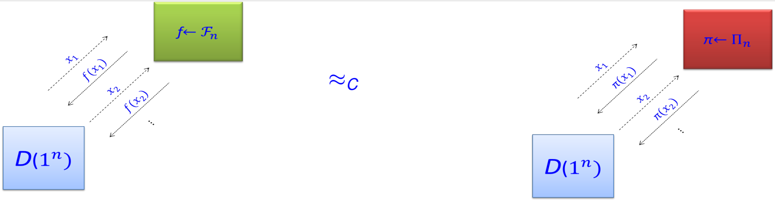

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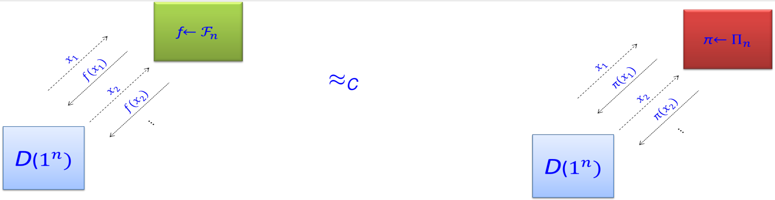

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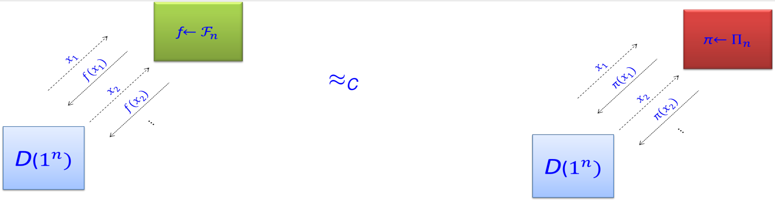

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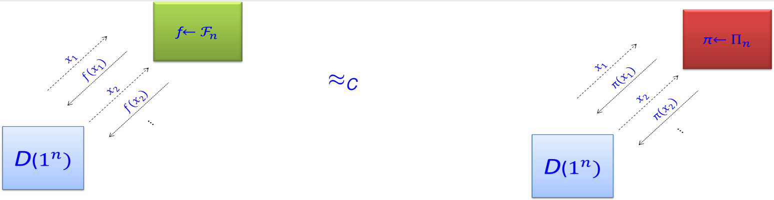

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- Main application: design a scheme assuming that you have random functions, and the realize them using PRFs.

Section 3

Pseudorandom Functions from One-Way Functions

Naive Construction

Let $G\colon \{0,1\}^n \mapsto \{0,1\}^{2n}$, and for $s \in \{0,1\}^n$ define $f_s\colon \{0,1\} \mapsto \{0,1\}^n$ by

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- Problem, we are constructing the whole truth table, even to compute a single output

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Construction 6 (GGM)

For $G\colon \{0,1\}^n \mapsto \{0,1\}^{2n}$ and $s \in \{0,1\}^n$,

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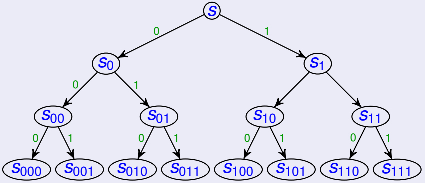

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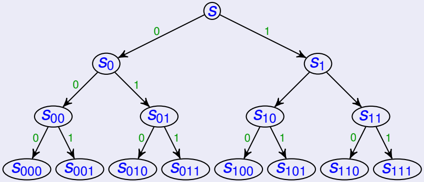

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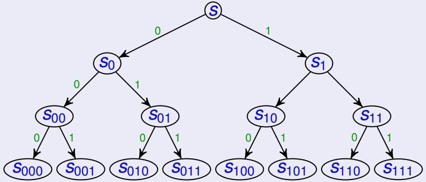

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- Example: $f_s(001) = s_{001} = G_1(s_{00}) = G_1(G_0(s_0)) = G_1(G_0(G_0(s)))$
- G is poly-time $\implies \mathcal{F} := \{\mathcal{F}_n = \{f_s\colon s \in \{0,1\}^n\}\}$ is efficient

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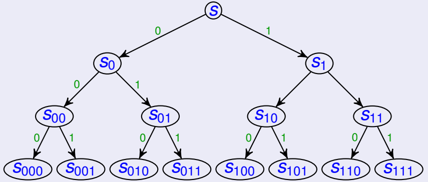

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- Example: $f_s(001) = s_{001} = G_1(s_{00}) = G_1(G_0(s_0)) = G_1(G_0(G_0(s)))$
- G is poly-time $\implies \mathcal{F} := \{\mathcal{F}_n = \{f_s \colon s \in \{0,1\}^n\}\}$ is efficient

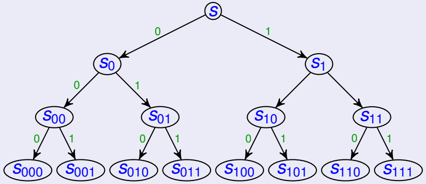
The GGM Construction

Construction 6 (GGM)

For $G\colon \{0,1\}^n \mapsto \{0,1\}^{2n}$ and $s \in \{0,1\}^n$,

- $G_0(s) = G(s)_{1,\ldots,n}$
- $G_1(s) = G(s)_{n+1,\ldots,2n}$

For $x \in \{0,1\}^k$ let $f_s(x) = G_{x_k}(f_s(x_{1,\ldots,k-1}))$, letting $f_s() = s$.



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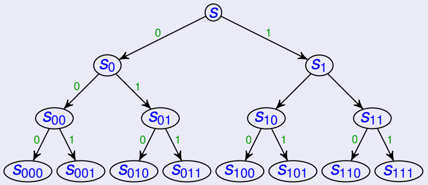

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Corollary 8

OWFs imply PRFs.

Proof Idea

Assume $\exists$ PPT D, $p \in$ poly and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\left|\Pr[\mathsf{D}^{F_n}(1^n) = 1] - \Pr[\mathsf{D}^{\Pi_n}(1^n) = 1]\right| \geq \frac{1}{p(n)}, \tag{1}$$

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Fix $n \in \mathbb{N}$ and let $t = t(n)$ be a bound on the running time of $\mathsf{D}(1^n)$. We use D to construct a PPT D′ such that

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where $(U_{2n})^t = U_{2n}^{(1)}, \ldots, U_{2n}^{(t)}$ and $G(U_n)^t = G(U_n^{(1)}), \ldots, G(U_n^{(t)})$.

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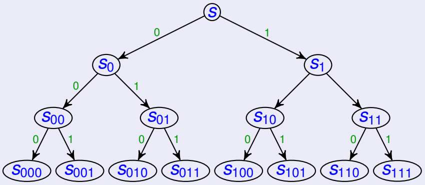
Hence, D′ violates the security of G

The Hybrid



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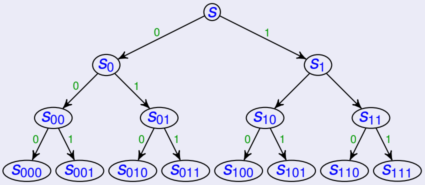

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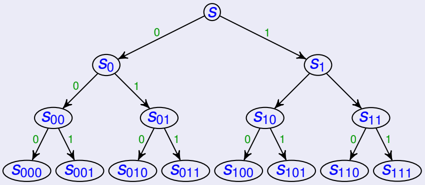

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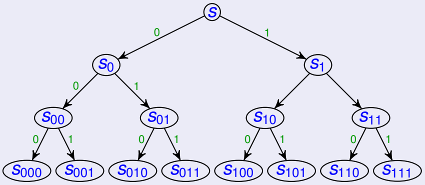

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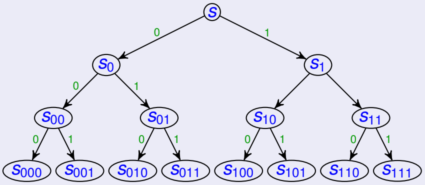

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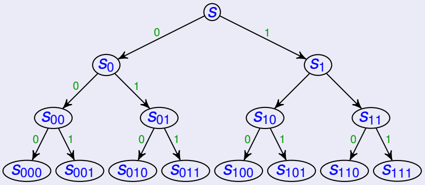

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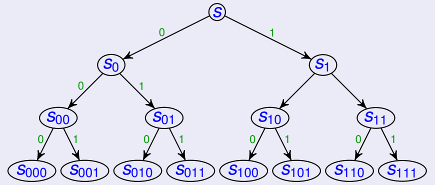

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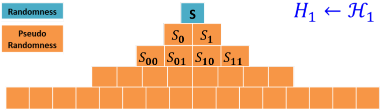

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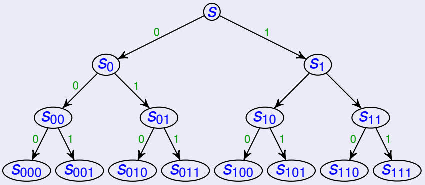

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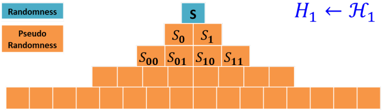

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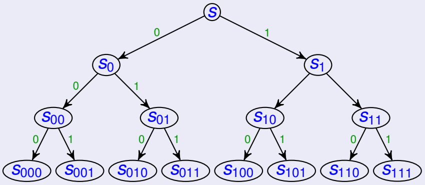

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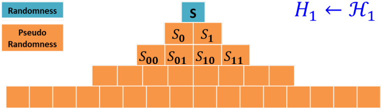

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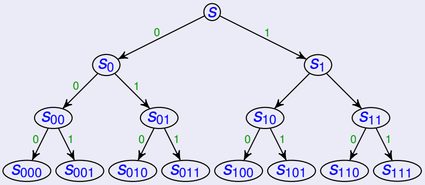

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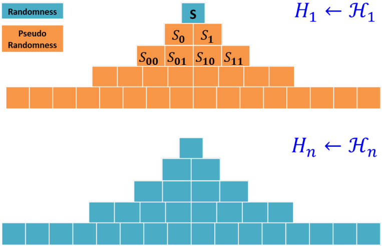

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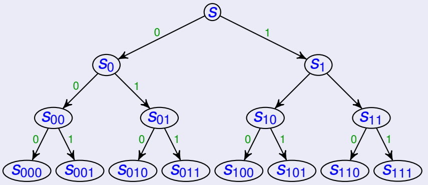

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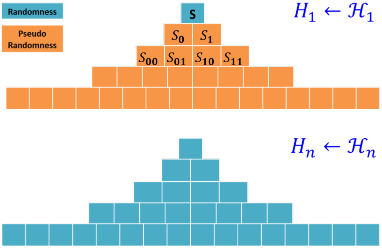

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- For some $i \in \{1,\ldots,i-1\}$, algorithm D distinguishes $\mathcal{H}_i$ from $\mathcal{H}_{i+1}$ by $\frac{1}{np(n)}$

The Hybrid cont.

We assume wlg. that D distinguishes between $\mathcal{H}_{n-1}$ and $\mathcal{H}_n$ (can we?)

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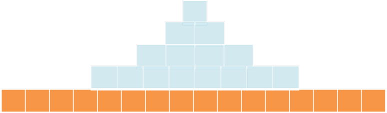
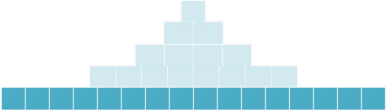


$\not\approx$



- D distinguishes (via t samples) between
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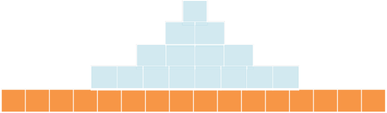

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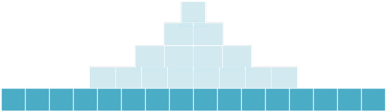

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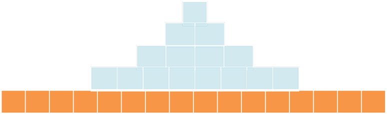

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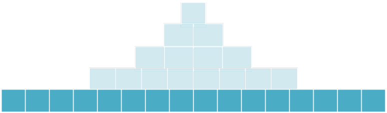

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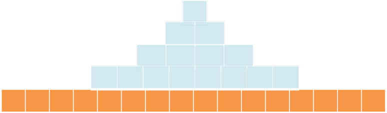

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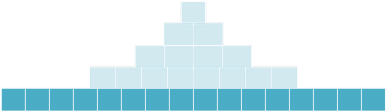

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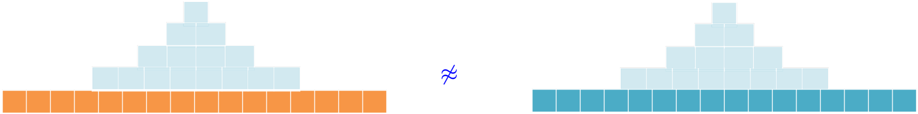

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Algorithm 9 (D′ on $y_1, \ldots, y_t \in (\{0,1\}^{2n})^t$)

Emulate D. On the i'th query q_i made by D:

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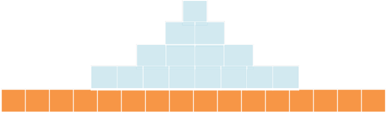

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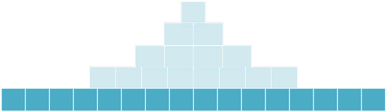

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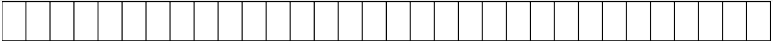

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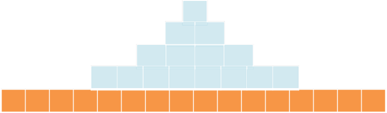

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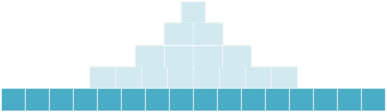

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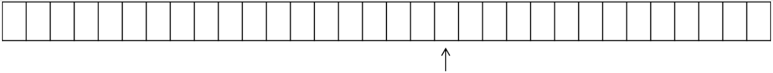

q_1

The Hybrid cont.

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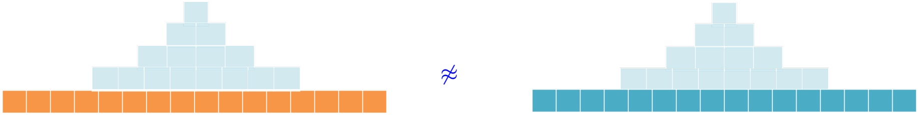

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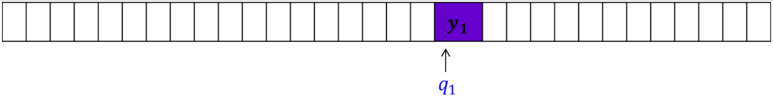

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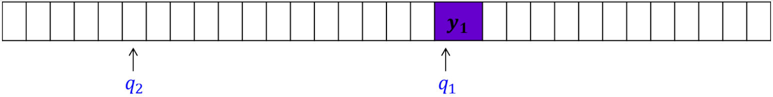

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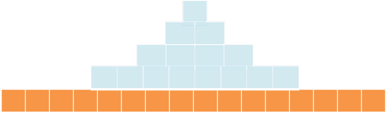

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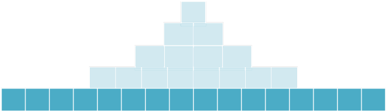

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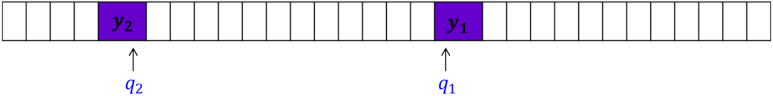

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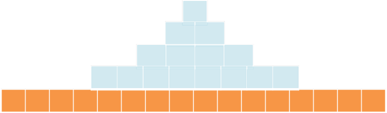

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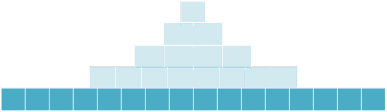

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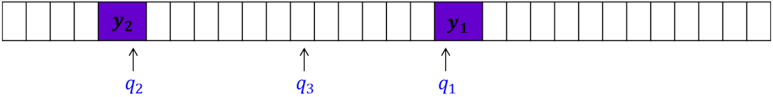

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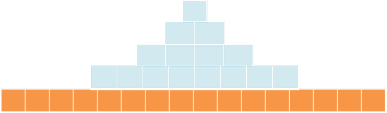

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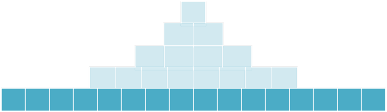

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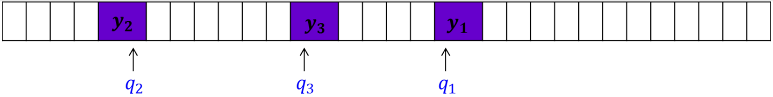

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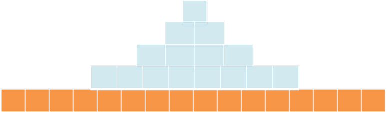

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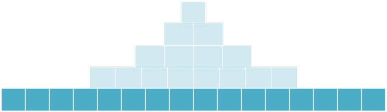

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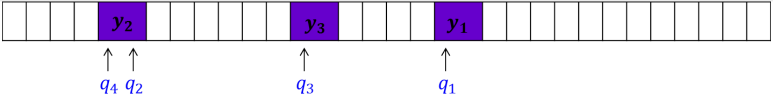

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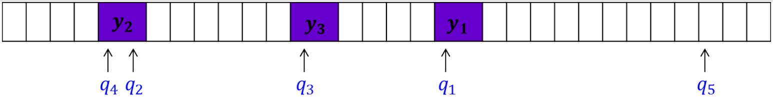

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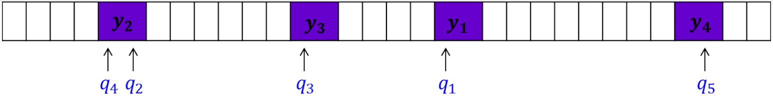

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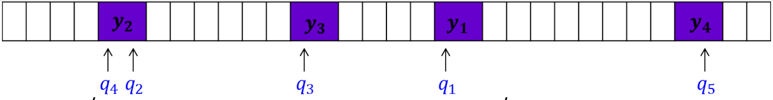

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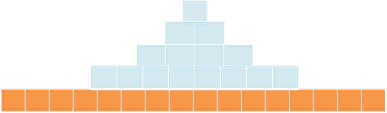

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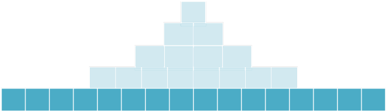

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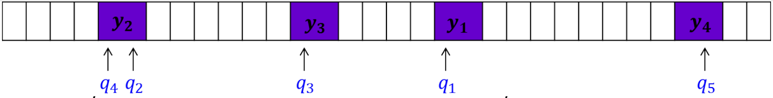

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- Hence, $\left|\Pr[D'((U_{2n})^t) = 1] - \Pr[D'(G(U_n))^t) = 1\right| > \frac{1}{np(n)}$

Part II

Pseudorandom Permutations

Formal Definition

Let $\widetilde{\Pi}_n$ be the set of all permutations over $\{0,1\}^n$.

Definition 10 (pseudorandom permutations (PRPs))

A permutation ensemble $\mathcal{F} = \{\mathcal{F}_n : \{0,1\}^n \mapsto \{0,1\}^n\}$ is a pseudorandom permutation, if

$$\left|\Pr[\mathsf{D}^{\mathcal{F}_n}(1^n) = 1] - \Pr[\mathsf{D}^{\widetilde{\Pi}_n}(1^n) = 1\right| = \operatorname{neg}(n), \tag{2}$$

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Section 4

PRP from PRF

Feistel Permutation

How does one turn a function into a permutation?

Feistel Permutation

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Definition 11 (LR)

For $f \colon \{0,1\}^n \mapsto \{0,1\}^n$, let $\mathsf{LR}_f \colon \{0,1\}^{2n} \mapsto \{0,1\}^{2n}$ be defined by

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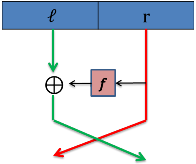

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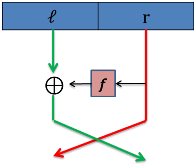

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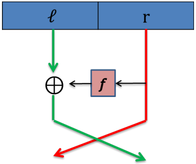

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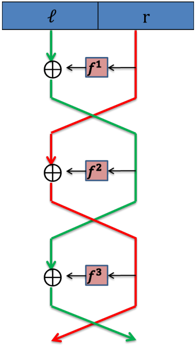

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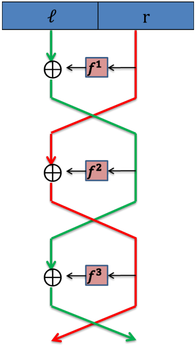

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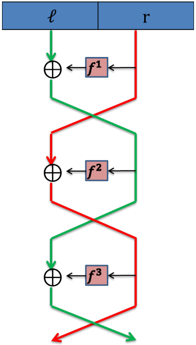

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  (letting LR_γ be the identity function)

Luby-Rackoff Thm.

Recall $\mathrm{LR}_f(\ell, r) = (r, f(r) \oplus \ell)$.

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Assuming that $\mathcal{F}$ is a PRF, then $\mathsf{LR}^3_{\mathcal{F}}$ is a PRP

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- To do that, we show both distributions are $O(q^2/2^n)$ close to $\mathit{Distinct} := \left((z_1, \ldots z_q) \xleftarrow{R} (\{0,1\}^{2n})^q \mid \forall i \neq j \colon (z_i)_0 \neq (z_j)_0\right)$.

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The statistical distance between distributions P and Q, is defined by

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Fact 17

Assume $\mathsf{SD}(P|\neg E, Q) \leq \delta_1$ *and* $\Pr_P[E] \leq \delta_2$, *then* $\mathsf{SD}(P,Q) \leq \delta_1 + \delta_2$

$(f(x_0), \ldots, f(x_q))_{f \xleftarrow{R} \widetilde{\Pi}}$ is close to *Distinct*

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ℓ_1^0	r_1^0	ℓ_2^0	r_2^0	$\ldots$	ℓ_q^0	r_q^0
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ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
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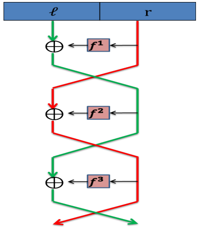
where $\ell_b^j = r_b^{j-1}$ and $r_b^j = f^j(r_b^{j-1}) \oplus \ell_b^{j-1}$.



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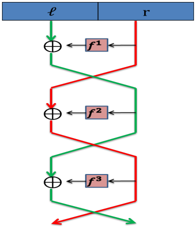

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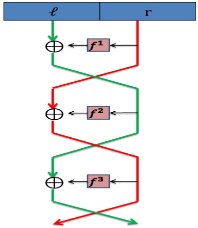

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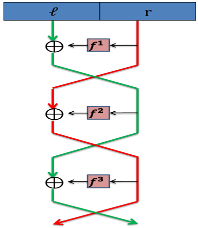
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
ℓ_1^3	r_1^3	ℓ_2^3	r_2^0	$\ldots$	ℓ_q^3	r_q^3

where $\ell_b^j = r_b^{j-1}$ and $r_b^j = f^j(r_b^{j-1}) \oplus \ell_b^{j-1}$.



Claim 20

$$\Pr_{f^1\xleftarrow{R}\Pi_n}\left[\mathrm{Bad}^1 := \exists i \neq j\colon r_i^1 = r_j^1\right] \leq \frac{\binom{q}{2}}{2^n}$$

Proof: $r_i^0 = r_j^0 \implies r_i^1 \neq r_j^1$

$(f(x_0),\ldots,f(x_q))_{f \xleftarrow{R} \mathrm{LR}^3(\Pi_n)}$ is close to *Distinct*

Let $(\ell_1^0, r_1^0), \ldots, (\ell_q^0, r_q^0) = (x_1, \ldots, x_k)$.

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ℓ_1^1	r_1^1	ℓ_2^1	r_2^1	$\ldots$	ℓ_q^1	r_q^1
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
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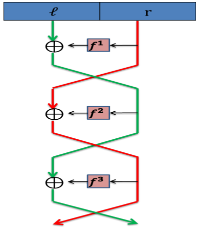

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ℓ_1^0	r_1^0	ℓ_2^0	r_2^0	$\ldots$	ℓ_q^0	r_q^0
ℓ_1^1	r_1^1	ℓ_2^1	r_2^1	$\ldots$	ℓ_q^1	r_q^1
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
ℓ_1^3	r_1^3	ℓ_2^3	r_2^0	$\ldots$	ℓ_q^3	r_q^3

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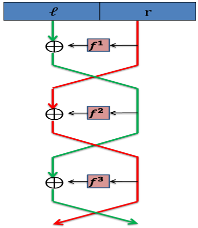

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$$\Pr_{(f^1,f^2) \xleftarrow{R} \Pi_n^2}\left[\mathrm{Bad}^2 := \exists i \neq j \colon r_i^2 = r_j^2\right] \leq 2 \cdot \frac{\binom{q}{2}}{2^n} \in O\left(\frac{q^2}{2^n}\right)$$

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ℓ_1^0	r_1^0	ℓ_2^0	r_2^0	$\ldots$	ℓ_q^0	r_q^0
ℓ_1^1	r_1^1	ℓ_2^1	r_2^1	$\ldots$	ℓ_q^1	r_q^1
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
ℓ_1^3	r_1^3	ℓ_2^3	r_2^0	$\ldots$	ℓ_q^3	r_q^3

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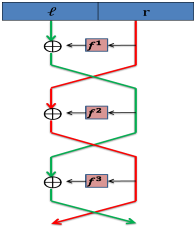

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$$\Pr_{f^1 \xleftarrow{R} \Pi_n}\left[\mathrm{Bad}^1 := \exists i \neq j \colon r_i^1 = r_j^1\right] \leq \frac{\binom{q}{2}}{2^n}$$

Proof: $r_i^0 = r_j^0 \implies r_i^1 \neq r_j^1$ and $r_i^0 \neq r_j^0 \implies \Pr_{f^1}\left[r_i^1 = r_j^1\right] = 2^{-n}$ $\square$

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$$\Pr_{(f^1,f^2) \xleftarrow{R} \Pi_n^2}\left[\mathrm{Bad}^2 := \exists i \neq j \colon r_i^2 = r_j^2\right] \leq 2 \cdot \frac{\binom{q}{2}}{2^n} \in O(\tfrac{q^2}{2^n})$$

Proof:

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ℓ_1^0	r_1^0	ℓ_2^0	r_2^0	$\ldots$	ℓ_q^0	r_q^0
ℓ_1^1	r_1^1	ℓ_2^1	r_2^1	$\ldots$	ℓ_q^1	r_q^1
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
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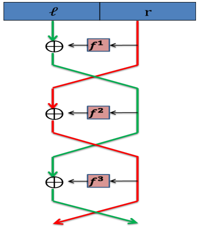

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$$\Pr_{f^1 \overset{R}{\leftarrow} \Pi_n}\left[\mathrm{Bad}^1 := \exists i \neq j \colon r_i^1 = r_j^1\right] \leq \frac{\binom{q}{2}}{2^n}$$

Proof: $r_i^0 = r_j^0 \implies r_i^1 \neq r_j^1$ and $r_i^0 \neq r_j^0 \implies \Pr_{f^1}\left[r_i^1 = r_j^1\right] = 2^{-n}$ $\square$

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ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
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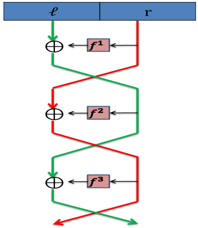

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Proof: similar to the above

Claim 22

$$\left((\ell_1^3, r_1^3),\ldots,(\ell_q^3, r_q^3) \mid \neg\mathrm{Bad}^2\right) \equiv \textit{Distinct}$$

$\left(f(x_0),\ldots,f(x_q)\right)_{f \xleftarrow{R} \mathrm{LR}^3(\Pi_n)}$ is close to *Distinct*

Let $(\ell_1^0, r_1^0),\ldots,(\ell_q^0, r_q^0) = (x_1,\ldots,x_k)$.

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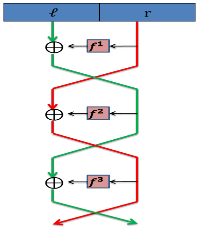
ℓ_1^1	r_1^1	ℓ_2^1	r_2^1	$\ldots$	ℓ_q^1	r_q^1
ℓ_1^2	r_1^2	ℓ_2^2	r_2^0	$\ldots$	ℓ_q^2	r_q^2
ℓ_1^3	r_1^3	ℓ_2^3	r_2^0	$\ldots$	ℓ_q^3	r_q^3



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Proof: ?

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  or at least, a less adaptive one